Consider,
\[ m(t) = A_m \cos \omega_m t \]

and recall that
\[ \theta_i(t) = \omega_i t + k_f \int_0^t m(\alpha) d\alpha \]

\[ \omega_i(t) = \frac{d \theta_i(t)}{dt} = \omega_c + k_f m(t) \]

Here,
\[ \omega_i(t) = \omega_c + k_f A_m \cos \omega_m t \]

\[ \omega_{i_{\text{max}}} = \omega_c + \frac{k_f A_m}{2} \Delta \omega \]

\[ \omega_{i_{\text{min}}} = \omega_c - \frac{k_f A_m}{2} \Delta \omega \]

\[ \omega_i(t) = \omega_c + \Delta \omega \cos \omega_m t \]

\[ \theta_i(t) = \int_0^t \omega_i(\tau) d\tau = \omega_c t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t + \theta_0 \]
\[ g_{FM}(t) = A \cos \left( \omega_c t + \frac{\Delta \omega}{\omega_m} \sin \omega_m t \right) \]
\[ \beta = \text{modulation index} \]

therefore
\[ g_{FM}(t) = A \cos \left( \omega_c t + \beta \sin \omega_m t \right) \]
\[ = \cos(\omega_c t) \cos(\beta \sin \omega_m t) - \sin(\omega_c t) \sin(\beta \sin \omega_m t) \]

when \( \beta \) is sufficiently small \((<< \frac{\pi}{2})\), we have
\[ \cos(\beta \sin \omega_m t) \approx 1 \]
\[ \sin(\beta \sin \omega_m t) \approx \beta \sin(\omega_m t) \]

\[ (\ast) \]

therefore
\[ g_{FM}(t) \approx \cos(\omega_c t) - \sin(\omega_c t) \left[ \beta \sin(\omega_m t) \right] \]
\[ = \frac{1}{2j} \left[ \delta(\omega - \omega_c - \omega_m) + \delta(\omega + \omega_c + \omega_m) \right] \]
\[ + \frac{\beta \pi}{2} \left[ \delta(\omega - \omega_c - \omega_m) - \delta(\omega - \omega_c + \omega_m) \right] \]
\[ - \frac{\beta \pi}{2} \left[ \delta(\omega + \omega_c - \omega_m) - \delta(\omega + \omega_c + \omega_m) \right] \]
\[ B_W = \omega w \]
Wideband FM

If \(|k_c a(t)| >> 1\), then the analysis of FM signals becomes very involved for a general modulating signal \(m(t)\).

We know that

\[
\omega_i(t) = \omega_c + k_{m} m(t)
\]

If

\[-m_p < m(t) < m_p\]

then

\[
\omega_c - k_{m} m \leq \omega_i(t) \leq \omega_c + k_{m} m
\]

Because \(\omega_i(t)\) varies in this range, we may be justified in assuming that the modulated-wave spectrum lies more or less in this range. If this is the case, the estimated BW, \(B_{FM}'\), is

\[
2\pi B_{FM}' \approx 2k_{m} m
\]

Define

\[
\Delta \omega = 2\pi f - k_{m} m
\]

then

\[
2\pi B_{FM} \approx 2(k_{m} m) = 2(2\pi \Delta f)
\]

\[
= 2\pi (2\Delta f)
\]

Hence

\[
B_{FM} \approx 2\Delta f
\]

where \(\Delta f\) is the maximum deviation of the carrier frequency \(f_c\).

We shall soon see that this expression is valid only when \(\Delta f >> B\). In the case \(\Delta f < B\), we have a case of NBFM where \(B_{FM} \approx 2B\) not \(2\Delta f\).
Spectrum of an Angle-Modulated Signal

The derivation of the spectrum of an angle-modulated signal is typically a very difficult task. However,  the problem is rather simple for tone modulation. Let

\[ m(t) = \alpha \cos \omega_m t \]

Recall that

\[ a(t) = \int_{-\infty}^{t} m(\alpha) d\alpha = (\alpha/\omega_m) \sin \omega_m t \]

Thus,

\[ \hat{\varphi}_{FM}(t) = A \exp \left[ j(\omega_c t + k_f a(t)) \right] \]

\[ = A \exp \left[ j(\omega_c t + k_f (\alpha/\omega_m) \sin \omega_m t \right] \]

Also recall that

\[ \Delta \omega = k_p m - k_f \alpha \]

For convenience, we define a deviation ratio \( \beta \) as

\[ \beta = \Delta \omega/\omega_m \]

Hence

\[ \beta = \Delta \omega/\omega_m = k_f \alpha/\omega_m \]
and

$$\hat{\varphi}_{FM}(t) = A \exp \left[ j[\omega_c t + k_z(\alpha/\omega) \sin \omega_m t] \right]$$

$$= A \exp \left[ j[\omega_c t + \beta \sin \omega_m t] \right]$$

$$= A \exp \left( j[\omega_c t] \exp[j \beta \sin \omega_m t] \right)$$

The term $\exp[j \beta \sin \omega_m t]$ is a periodic signal with period $2\pi/\omega_m$ and can be expanded by exponential Fourier series:

$$\exp[j \beta \sin \omega_m t] = \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j n \omega_m t)$$

where the Fourier coefficients are

$$J_n(\beta) = \frac{1}{2\pi/\omega_m} \int_{-\pi/\omega_m}^{\pi/\omega_m} \exp[j \beta \sin \omega_m t] \exp(-j n \omega_m t) \, dt$$
Letting $\omega_m = x$, we get

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j\beta \sin x] \exp(-j n x) \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - n x)] \, dx$$

The function $J_n(\beta)$ cannot be evaluated in a closed form and is referred to as the Bessel function of the first kind and nth order. The following figure shows a plot of this function.

**FIGURE 3.29** $J_n(\beta)$ as a function of $\beta$
Figure 4.53 (a) Variations of $J_n(\beta)$ as a function of $n$ for various values of $\beta$. (b) Tone-modulated FM wave spectrum.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$J_0(\beta) = 0$</th>
<th>$\beta_{n0}$</th>
<th>$\beta_{n1}$</th>
<th>$\beta_{n2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$J_0(\beta) = 0$</td>
<td>2.4048</td>
<td>5.5201</td>
<td>8.6537</td>
</tr>
<tr>
<td>1</td>
<td>$J_1(\beta) = 0$</td>
<td>0.0000</td>
<td>3.8317</td>
<td>7.0156</td>
</tr>
<tr>
<td>2</td>
<td>$J_2(\beta) = 0$</td>
<td>0.0000</td>
<td>5.1355</td>
<td>8.4172</td>
</tr>
<tr>
<td>4</td>
<td>$J_4(\beta) = 0$</td>
<td>0.0000</td>
<td>7.5883</td>
<td>—</td>
</tr>
<tr>
<td>6</td>
<td>$J_6(\beta) = 0$</td>
<td>0.0000</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
Recall that

\[ \varphi_{\text{FM}}(t) = A \exp\left[j(\omega_c t + \beta \sin \omega_m t)\right] \]

\[ = A \exp[j(\omega_c t)] \exp[j\beta \sin \omega_m t] \]

Hence

\[ \varphi_{\text{FM}}(t) = A \exp[j(\omega_c t)] \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j n\omega_m t) \]

\[ = A \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j(\omega_c t + n\omega_m t)) \]

and

\[ \varphi_{\text{FM}}(t) = \text{Re}\{ \varphi_{\text{FM}}(t) \} = \]

\[ = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c t + n\omega_m t) \]

The modulated signal has a carrier component and an infinite number of sidebands of frequencies \( \omega_c \pm \omega_m, \omega_c \pm 2\omega_m, \ldots, \omega_c \pm n\omega_m, \ldots \). The strength of the nth sideband at \( \omega_c \pm n\omega_m \) is \( J_n(\beta) \).
Remarks

1. It is seen from the figure of $J_n(\beta)$ that for given $\beta$, $J_n(\beta)$ decreases with $n$. For large $n$, therefore, $J_n(\beta)$ is negligible and the number of significant sidebands are finite.

2. It is observed that $J_n(\beta)$ is negligible for $n > \beta + 2$. Hence the number of significant sidebands is $\beta + 2$. Thus

$$\varphi_{FM}(t) = A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

$$\equiv A \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

The BW, $2\pi B_{FM}$, is

$$2\pi B_{FM} = [\omega_c + (\beta+2)\omega_m] - [\omega_c - (\beta+2)\omega_m] = 2(\beta+2)\omega_m$$

Recall that

$$\beta = \Delta\omega/\omega_m$$

Hence

$$2\pi B_{FM} = 2(\beta+2)\omega_m = 2(\beta\omega_m + 2\omega_m)$$

$$= 2(\Delta\omega + 2\omega_m)$$

$$= 2[2\pi\Delta f + 2(2\pi f_m)]$$

That is

$$2\pi B_{FM} = 2[2\pi\Delta f + 2(2\pi f_m)]$$

or

$$B_{FM} = 2(\Delta f + 2f_m) = 2(\Delta f + 2B)$$
3. It is seen from the graph of the Bessel function that for $\beta \ll 1$, there are only two significant terms corresponding to $n = 0$ (carrier) and $n = 1$; therefore, there exist only one significant sideband. Hence,

$$B_{FM} = 2f_m = 2B$$

which verifies our previous result on NBFM.
4. The results of BW so far are for tone modulation. For a general case, it can be shown that

\[ B_{FM} = 2(\Delta f + 2B) \]  \hspace{1cm} (*)

In the literature, other rule of thumb for the FM bandwidth are found. The most commonly used rule, known as carson's rule, is

\[ B_{FM} = 2(\Delta f + B) \]

This rule gives a better BW estimate than (*) in the narrowband case, where

\[ \Delta f \ll B \]

and

\[ B_{FM} \geq 2B \]

If \( \Delta f \ll B \) is not satisfied (wideband and intermediate cases), (*) gives a better estimate.

For the general modulation case, the deviation ratio \( \beta \) is defined as

\[ \beta = \Delta f / B \]

Hence

\[ B_{FM} = 2(\Delta f + kB) = 2(\beta B + kB) = 2B (\beta + k) \]

where \( 1 < k < 2 \) to satisfy both BW rules.
Wideband PM

All the results derived for FM can be directly applied to PM. Recall that for PM we have

$$\varphi_{pM}(t) = A \cos (\omega_c t + k m(t))$$

where the instantaneous frequency $$\omega_1(t)$$ is given by

$$\omega_1(t) = d\theta(t)/dt = \omega_c + k_p [dm(t)/dt]$$

Thus, for PM

$$\Delta \omega = k_p [dm(t)/dt]_{max} = k m'_{pP}$$

where

$$m'_{pP} = [dm(t)/dt]_{max}$$

and

$$B_{pM} = 2(\Delta f + kB)$$

$$= 2[k_p /2\pi]m'_{pP} + kB$$

Remarks

One important difference exists between FM and PM with regard to $$\Delta f$$. In FM, $$\Delta \omega = k m_{pP}$$ depends only on the peak value of $$m(t)$$. It is independent of the spectrum of $$m(t)$$. On the other hand, in PM, $$\Delta \omega = k m'_{pP}$$ depends on the peak value of $$dm(t)/dt$$ which depends strongly on the frequency spectrum of $$m(t)$$. The presence of higher-frequency components in $$m(t)$$ causes rapid variations, resulting in a higher value of $$m'_{pP}$$. Similarly, predominance of lower-frequency components will result in a lower value of $$m'_{pP}$$. Hence, whereas the WBFM carrier $$BW$$ is practically independent of the spectrum of $$m(t)$$, the WBPM carrier $$BW$$ strongly depends on the spectrum of $$m(t)$$. 
These conclusions can be verified for tone modulation, where

\[ m(t) = \alpha \cos \omega t \]

\[ \frac{dm(t)}{dt} = -\alpha \omega \sin \omega t \]

Hence,

\[ (\Delta \omega)_M = k \frac{m}{f} = k \alpha \]

\[ (\Delta \omega)'_M = k \frac{m'}{f} = k \alpha \omega \]
For FM we have

\[
\omega_i(t) = \omega_c + k_f m(t) = 10^8 + 10^5 m(t)
\]

\[
(\omega_i)_{\text{max}} = 10^8 + 10^5
\]

\[
(\omega_i)_{\text{min}} = 10^8 - 10^5
\]

\[
\Delta \omega = k_f m_p = 10^5
\]
(6) For 

\[ w_i(t) = w_c + k_p \frac{d w_i(t) }{\sqrt{t}} = 10^8 + 25 \frac{d w_i(t)}{\sqrt{t}} \]
\[
\begin{align*}
(\omega_i)_{\text{max}} &= 10^8 + (25)(8000) \times 10^5 \\
(\omega_i)_{\text{mic}} &= 10^8 - (25)(8000) \times 10^5 \\
\Delta \omega &= (25)(8000) = 2 \times 10^5
\end{align*}
\]
\[ \Delta \omega = K_f M_p = 10^{10} \text{ Hz} \]
\[ (\Delta \omega)_\text{max} = 10^4 + 10^6 \text{ Hz} \]
\[ \Phi_{FM}(t) \]

\[ \Delta \omega = K \omega M_p' = 2 \times 10^4 \text{ Hz} \]
\[ (\Delta \omega)_\text{max} = 10^4 + 2 \times 10^4 \text{ Hz} \]
\[ \Phi_{FM}(t) \]

\[ \Delta \omega = K_f M_p = 2000 \text{ Hz} \]
\[ (\Delta \omega)_\text{max} = 10^6 + 2000 \text{ Hz} \]
\[ (\Delta \omega)_\text{min} = 10^6 - 2000 \text{ Hz} \]

\[ M(t) = \begin{cases} 1 + 1000t & 0 < t < 10^{-3} \\ -3 + 1000t & 10^{-3} < t < 2 \times 10^{-3} \end{cases} \]

\[ \Phi_{FM}(t) = A \cos \left[ 10^6 t + K \omega M(t) \right] \]
\[ \Phi_{pm}(t) = \begin{cases} A \cos \left[ 10^6 t + \frac{\pi}{4} (1 + 10000 t) \right] & ; 0 < t < 10^{-3} \\ A \cos \left[ 10^6 t + \frac{\pi}{4} (2 + 10000 t) \right] & ; 10^{-3} < t < 2 \times 10^{-3} \end{cases} \]

\[ W(t) = \frac{d\Phi(t)}{dt} \]

\[ W_{i}(t) = \begin{cases} 10^6 + 250 \pi & ; 0 < t < 10^{-3} \\ 10^6 + 250 \pi & ; 10^{-3} < t < 2 \times 10^{-3} \end{cases} \]

Therefore, for all \( t \), \( \Phi_{pm}(t) \) has the same frequency, that is, \( 10^6 + 250 \pi \). However, at jump discontinuities in \( m(t) \), there will be abrupt phase changes of \( \pi \).

\[ \frac{d}{dt} [m(0^+) - m(0^-)] = \frac{\pi}{2} \times 2 = \frac{\pi}{2} \]

Hence at \( t = 0 \), phase increases instantaneously by \( \frac{\pi}{2} \) and at \( t = 10^{-3} \) the phase decreases instantaneously by \( \frac{\pi}{2} \) as shown in the figure below.

\[ \frac{\pi}{4} - \left( \frac{-3n}{4} \right) = n \]
Because there are jump discontinuities, it is easier to use the direct approach to PM.

\[ \phi_{pm}(t) = A \cos \left( \omega_c t + k_p m(t) \right) = A \cos \left( \omega_c t + \frac{\pi}{2} m(t) \right) \]

For the first cycle (see adjacent figure), \( m(t) = 2000 \) and (for the first cycle)

\[ \phi_{pm}(t) = A \cos \left( \omega_c t + 1000\pi t \right) \]

For the second cycle,

\[ m(t) = -2 + 2000 \]

\[ \phi_{pm}(t) = A \cos \left[ \left( \omega_c + 1000\pi \right) t - \pi \right] \]

Hence there is a phase discontinuity of \(-\pi\) at the end of each cycle (see adjacent figure). Because of jump discontinuities in \( m(t) \), \( |k_p m(t)| < \frac{\pi}{2} \) for unambiguous demodulation or \( k_p < \frac{\pi}{2} \).

\[ w_c(t) = \frac{d}{dt} \left( \theta(t) \right) \]

\[ = w_c + 1000 \]
For FM \((\omega_c)_{\text{min}} = 10^6 \times 1000, (\omega_c)_{\text{max}} = 10^6 + 1000\)

For the interval \(a\) (the half cycle of the modulating signal), the carrier frequency increases linearly from \((\omega_c)_{\text{min}}\) to \((\omega_c)_{\text{max}}\), then stays at \((\omega_c)_{\text{max}}\) for the next half cycle (interval \(a\)) and repeats like this indefinitely.

For PM
\[(\omega_c)_{\text{max}} = \omega_c + K_p m' = 10^6 + 2000\pi = 10^6 + 2000\pi\]
\[(\omega_c)_{\text{min}} = 10^6 - 2000\pi\]

For the interval \(a\) (the half cycle of \(m(t)\)), where \(m(t)\) is increasing linearly, the carrier frequency is a constant \((\omega_c)_{\text{max}} = 10^6 + 2000\pi\). For the next half cycle, the carrier frequency is \(\omega_c = 10^6\). At the discontinuity of \(m(t)\), there is a phase discontinuity of \(K_p\) (the amplitude discontinuity of \(m(t)\)) = \(2\pi\).

\[\Phi_{\text{PM}}(t) = \frac{\pi}{2} = \pi \text{ rad.}\]
4.13-1 \[ m(t) = \sin 30\pi t + 3\cos 200t \]

\[ \Phi_{pm}(t) = A \cos \left[ 10^7 t + 10 (\sin 30\pi t + 3\cos 200t) \right] \]

Also, \[ \int_0^t m(t) \, dt = -\frac{1}{30\pi} \cos 30\pi t + \frac{3}{200} \sin 200t \]

and

\[ \Phi_{FM}(t) = A \cos \left[ 10^7 t - \frac{1}{3\pi} \cos 30\pi t + \frac{3}{200} \sin 200t \right] \]

For \( PM \), \( \Delta \omega = K_p m_p = 10(30\pi \cos 30\pi t + 600\sin 200t) \max \)

Assuming that peaks \( \cos 30\pi t \) and \( \sin 200t \) add in phase at some instant,

\[ m_p = 30\pi t + 600 = 694.25 \]

and

\[ \Delta \omega = 10 \times 694.25 = 6942.5 \]

\[ \Delta f = \frac{\Delta \omega}{2\pi} = 1104.9 \text{ Hz} \]

The signal bandwidth \( B = \frac{600}{2\pi} = 95.5 \text{ Hz} \)

This is a wideband case. Hence

\[ B_{PM} = 2(\Delta f + 2B) = 2(1104.9 + 191) = 2591.8 \text{ Hz} \]

For \( FM \), \( \Delta \omega = K_f m_p = 10(1+3) = 40 \)

\[ \Delta f = \frac{40}{2\pi} = 6.37 \text{ Hz} \]

Signal bandwidth \( B = \frac{200}{2\pi} = 31.8 \gg \Delta f \)

This is a narrowband case, and

\[ B_{FM} = 2(\Delta f + B) = 203.74 \text{ Hz} \]

\[ = 2(6.37 + 31.8) = 76.34 \text{ Hz} \]
4.13-2 \[ E_m(t) = 10 \cos(w_c t - 0.3 \cos 200 \pi t) \]

(a) \[ \text{power} = A^2/2 = 50 \]

(b) \[ w_i = w_c + (0.3 \times 200 \pi \sin 200 \pi t) \max \]

\[ = w_c + 60 \pi \]

\[ \Delta w = 60 \pi \text{ and } \Delta f = 20 \]

(c) \[ \text{phase deviation } \Delta \phi = (-0.3 \cos 200 \pi t) \max = 0.3 \]

(d) \[ \text{Signal bandwidth } B = \frac{200 \pi}{2\pi} = 100 \text{ Hz and } \Delta f = 20 \text{ Hz} \]

This is a narrowband case. Hence \[ B_{EM} = 2(\Delta f + B) = 240 \text{ Hz} \]

4.13-3 (a) \[ \text{power} = 2^{3/2} = 2 \]

(b) \[ w_i = w_c + (15000 \cos 3000 t + 20000 \pi \cos 50 \pi t) \max \]

\[ = w_c + (15000 + 20000 \pi) = w_c + 77831.85 \]

\[ \Delta w = 77831.85 \text{ and } \Delta f = 12387.32 \text{ Hz} \]

(c) \[ \text{phase deviation } \Delta \phi = 5 + 10 = 15 \text{ rad} \]

(d) \[ \text{Signal bandwidth } B = \frac{20000 \pi}{2\pi} = 10000 \text{ Hz} \]

\[ \text{and } \Delta f = 12387.32 \text{ Hz} \]

This is an intermediate case (more of a wideband case). \[ B_{EM} = 2(\Delta f + B) = 644775 \text{ kHz} \]
\[
\Phi_{Em}(t) = 2 \cos \left(2\pi \times 10^7 t + 5 \sin 3000 t + 10 \sin 2000 n t \right)
\]

(a) \[\text{Power} = \frac{2^2}{1} = 2\]

(b) \[\theta(t) = 2\pi \times 10^7 t + 5 \sin 3000 t + 10 \sin 2000 n t\]

\[w_i(t) = \frac{d\theta(t)}{dt} = 2\pi \times 10^7 + 15000 \cos 3000 t + 20000 \cos 2000 n t\]

\[\Delta w = 15000 + 20000 n = 77,800\]

\[\therefore \Delta f = 12,388.53 \text{ Hz}\]

(c) \[\Delta \Phi = 5 + 10 = 15 \text{ rad}\]

(d) \[B = \frac{2000 n t}{2\pi} = 1000 \text{ Hz}.

\[\Delta f > B \Rightarrow \text{Wideband}\]

\[
B_{Em} = 2(\Delta f + 2B) = 2(12,388.53 + 2000) = 28,777 \text{ Hz}
\]
4.13-4 The bandwidth \( B = 5 \times 1000 = 5000 \) Hz

For FM, \( \Delta f = \frac{K_{m}P}{2T} = \frac{10^{5}/2\pi}{2\pi} \)

Hence \( \Delta f > B \), and we have WB FM

\[ B_{FM} = 2(\Delta f + 2B) = 2 \left( \frac{10^{5}/2\pi}{10000} \right) = 51.83 \text{ kHz} \]

For PM, \( \Delta f = \frac{K_{m}P}{2T} = \frac{10^{5}/2\pi}{2\pi} = 10^{5}/2\pi \)

Hence \( \Delta f \gg B \), and we have wideband case.

Therefore \( B_{PM} = 2(\Delta f + 2B) = 2 \left( \frac{10^{5}/2\pi}{10000} \right) = 83.66 \text{ kHz} \)

\[ \times \]

4.13-5 \( B = 3 \times 1000/2\pi = 150/\pi \) Hz

For FM: \( \Delta f = \frac{K_{m}P}{2\pi} = \frac{10^{5}/2\pi \times 1000}{2\pi} = 5 \)

Hence \( \Delta f < B \), and this is a narrowband case.

\[ B_{FM} = 2(\Delta f + B) = 2 \left( 5 + \frac{150}{\pi} \right) = 105.5 \text{ Hz} \]

For PM: \( \Delta f = \frac{K_{m}P}{2T} = \frac{10^{5}/2\pi \times 1000}{2\pi} = 500 \)

Hence \( \Delta f \gg B \), and we have a wideband case.

Therefore, \( B_{PM} = 2(\Delta f + 2B) = 2 \left( 500 + \frac{150}{\pi} \right) = 1099.5 \text{ Hz} \)

\[ \times \]

4.13-6 \( B = 5 \times 0.5 = 2.5 \text{ Hz} \)

For FM: \( \Delta f = \frac{K_{m}P}{2\pi} = \frac{10^{5}/2\pi \times 1000}{2\pi} = 500 \)

Hence \( \Delta f \gg B \), and we have a wideband case. Therefore

\[ B_{FM} = 2(\Delta f + 2B) = 2 \left( 500 + 5 \right) = 1010 \text{ Hz} \]

For PM: \( \Delta f = \frac{K_{m}P}{2T} = \frac{10^{5}/2\pi \times 2000}{2\pi} = 1000 \)

Therefore, \( \Delta f \gg B \), and this is a wideband case. Hence

\[ B_{PM} = 2(\Delta f + 2B) = 2 \left( 500 + 5 \right) = 2010 \text{ Hz} \]
4.13-7 \[ B = 7 \times 500 = 3500 \text{ Hz} \]

FM: \[ \Delta f = \frac{K_f M_p}{2\pi} = \frac{1000 \times 2}{2\pi} = \frac{1000}{\pi} \]

Therefore, \( \Delta f < B \) and we have a narrowband case. Hence

\[ B_{FM} = 2(\Delta f + B) = 2\left(\frac{1000}{\pi} + 3500\right) = 7636.6 \text{ Hz} \]

PM: We note from the solution in Prob. 4.12-4 that the PM waveform in this case is identical to the PSK waveform shown in Fig. 3.51b. Therefore, the PM waveform is identical to PSK corresponding to \( \hat{m}(t) \) shown in the adjacent figure.

\[ \phi_{PM}(t) = \hat{m}(t) \cos \omega_c t \]

Since the bandwidth of \( m(t) \) or \( \hat{m}(t) \) is 3500 Hz (7th harmonic),

\[ B_{PM} = 2 \times 3500 = 7000 \text{ Hz} \]
(a) For $g_1(t)$, $\Delta f = \frac{Kmf_{c}}{2\pi} = \frac{0.05 \times 200}{2\pi} = \frac{5}{\pi}$

Hence for $g_2(t)$, $\Delta f = 2500 \times \frac{5}{\pi} = 12500/\pi$ Hz

Bandwidth of baseband signal is 100 Hz.

Hence $g_2(t)$ is wideband and bandwidth of $g_2(t)$ is $2(\Delta f + 2B) = 2\left(\frac{12500}{\pi} + 200\right) = 8358$ Hz.

(b) The carrier frequency of $g_2(t)$ is $10^5 \times 250 = 250$ MHz

$g_2(t) = A\cos\left[2\pi t \times 250 \times 10^6 t + (0.05 \times 2500) \int m(\omega) \, d\omega\right]$

$= A\cos\left[5 \pi t \times 10^6 t + 125 \int m(\omega) \, d\omega\right]$

$\omega_i(t) = 5 \pi t \times 10^6 + 125 \int m(t) \, dt$

Hence

$\Delta \omega = 125f_{m} = 125 \times 200$

and $\Delta f = \frac{\Delta \omega}{2\pi} = 12500/\pi$

$B_{FM} = 2(\Delta f + 2B) = 2\left(\frac{12500}{\pi} + 200\right) = 8358$ Hz

This agrees with the result found in (a).

(c) The frequency converter shifts the carrier frequency from 250 MHz to 101 MHz. Hence

$g_3(t) = A\cos\left[2\pi t \times 101 \times 10^6 t + 125 \int m(\omega) \, d\omega\right]$

The frequency deviation and the bandwidth of $g_3(t)$ are identical to those of $g_2(t)$. 

4.14 - 2 For first stage \( f = 0.2 = \frac{Af}{50} \)

Hence \( Af = 10 \text{ Hz} \)

To get \( Af = 20000 \), we need frequency multiplication \( M = \frac{20000}{10} = 2000 \)

Since we can use only frequency doublers and triplers, we must use 11 doublers to give \( 2^n = 2048 \). Hence \( M = 2048 \). We must go back and adjust \( \beta \) to new value such that first stage \( Af = \frac{20000}{2048} = 9.766 \) and \( \beta = \frac{Af}{50} = 0.1953 \)

We can break this into two stages with \( M_1 = 64 \) and \( M_2 = 32 \) as shown in the figure below.

The carrier frequency after first multiplication \((M = 64)\) is \( 200 \times 10^3 \times 64 = 12.8 \text{ MHz} \). The final carrier frequency is 100 MHz. Hence the carrier frequency before the multiplication \( M_2 \) \((M_2 = 32)\) should be \((100/32) = 3.125 \text{ MHz}\)

Since the output of \( M_1 \) has a carrier frequency of 12.8 MHz, we need a converter whose local oscillator (crystal oscillator) frequency is \( (12.8 - 3.125) = 9.675 \text{ MHz} \)

\[
\begin{align*}
f_c &= 200 \text{ kHz} \\
Af &= 9.766 \\
f_{x64} &= 12.8 \text{ MHz} \\
\text{Af} &= 625.024 \\
f_{x32} &= 3.125 \text{ MHz} \\
\text{Af} &= 625.024 \\
f_3 &= 100 \text{ kHz} \\
\text{Af} &= 20 \text{ kHz} \\
f_3 &= 9.675 \text{ MHz}
\end{align*}
\]
(a) \[ \phi_{ph}(t) = A \cos \left[ \omega_c t + km(t) \right] \]

If we apply this to an FM demodulator, the demodulator output is
\[ e_1(t) = km(t) \]
when this is passed through an integrator, the integrator output \( e_0(t) \) is
\[ e_0(t) = \int e_1(k) \, dk = km(t) \]

Hence an FM demodulator followed by an integrator demodulates a FM wave. The only restriction is that \( m(t) \) cannot have jump discontinuity. If \( m(t) \) has jump discontinuities, \( m(t) \) will be infinite at the points of discontinuities and the system will not work.

(b) \[ \phi_{ph} = A \cos \left[ \omega_c t + k \int m(a) \, da \right] \]

If we apply \( \phi_{ph}(t) \) to a PM demodulator, the demodulator output is \( k \int m(a) \, da \). When this is passed through a differentiator, the differentiator output is \( km(t) \) regardless of whether \( m(t) \) has jump discontinuities or not.
(a) \( \phi_{\text{fm}}(t) = A \cos \left[ \omega_c t + K_p m(t) \right] \)
\( = A \cos \left[ \omega_c t + \frac{\pi}{2} m(t) \right] \)
\( = -A \sin \omega_c t \quad \text{for} \ m(t) = 1 \text{ and } -3 \)
\( = A \sin \omega_c t \quad \text{for} \ m(t) = -3 \text{ and } -1 \)

If \(-A \sin \omega_c t\) is received, \(m(t)\) could be 1 or -3 and
if \(A \sin \omega_c t\) is received, \(m(t)\) could be 3 or -1. It
is impossible to demodulate \(\phi_{\text{fm}}(t)\) unambiguously.

(b) In this case the phase of \(\phi_{\text{fm}}(t)\) changes by \(\pi\) radians
when \(m(t)\) changes from 1 to 3, or vice versa and
from 1 to -1 and vice versa. But the phase shift
is not instantaneous as seen from the figure below.
There is a phase continuity throughout. When
such a \(\phi_{\text{fm}}(t)\) is passed through an FM demodulat-
or, the demodulator output is \(m(t)\). When this
is integrated, we obtain \(m(t)\) without ambiguity.
CHAPTER SEVEN

Problem 7.1

a) \( y(t) = x(t) \cos(\omega_c t + \theta_c) \); \( w(t) = y(t) \cos(\omega_c t + \theta_c) = x(t) \cos^2(\omega_c t + \theta_c) \)

But \( \cos^2 \theta = \frac{1}{2} [1 + \cos 2\theta] \), so \( \cos^2(\omega_c t + \theta_c) = \frac{1}{2} [1 + \cos(2\omega_c t + 2\theta_c)] \) and \( w(t) = \frac{1}{2} x(t) + \frac{1}{2} x(t) \cos(2\omega_c t + 2\theta_c) \).

b) To gain insight, we sketch the following spectra.

Note that to avoid overlap between the two sidebands we must have \( \omega_c > \omega_m \). Finally, we sketch \( W(w) \):

Clearly, in order for the output to be proportional to \( x(t) \) we must have \( \omega_m < W < 2\omega_c - \omega_m \). This answer does not depend on \( \Theta_c \).

Problem 7.2

Sketches of each step toward \( \tilde{Y}(w) \) are given in order below.

Problem 7.3

a) \( w(t) = x(t) \cos(\omega_c t) \cos(\omega_2 t) \). But \( \cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)] \)

so \( w(t) = \frac{1}{2} x(t) [\cos(\Delta \omega t) + \cos(\omega_c + \omega_2 - \omega_2)t] \). But \( \frac{1}{2} x(t) \cos[(\omega_c + \omega_2)t] \)

has a spectrum in the range \( \omega_2 + \omega_2 - \omega_2 \leq W \leq \omega_c + \omega_2 + \omega_2 \), which can be rewritten as \( 2\omega_2 + (\omega_2 - \omega_2) - \omega_2 \leq W \leq 2\omega_2 + (\omega_2 - \omega_2) + \omega_2 \). Using \( \Delta \omega = \omega_2 - \omega_c \)

this becomes \( 2\omega_2 + \Delta \omega - \omega_2 \leq \omega \leq 2\omega_2 + \Delta \omega + \omega_2 \). However, we are given \( W < 2\omega_2 + \Delta \omega - \omega_2 \). So lowpass filtering output = \( \frac{1}{2} x(t) \cos(\Delta \omega t) \)