Stability Condition of an LTI Discrete-Time System

- **BIBO Stability Condition** - A discrete-time is BIBO stable if the output sequence $\{y[n]\}$ remains bounded for all bounded input sequence $\{x[n]\}$
- An LTI discrete-time system is BIBO stable if and only if its impulse response sequence $\{h[n]\}$ is absolutely summable, i.e.

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Stability Condition of an LTI Discrete-Time System

- Proof: Assume $h[n]$ is a real sequence
- Since the input sequence $x[n]$ is bounded we have

$$|x[n]| \le B_x < \infty$$

- Therefore

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$\le B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_x S$$

Stability Condition of an LTI Discrete-Time System

- Thus, $S < \infty$ implies $|y[n]| \le B_y < \infty$ indicating that $y[n]$ is also bounded
- To prove the converse, assume $y[n]$ is bounded, i.e., $|y[n]| \le B_y$
- Consider the input given by

$$x[n] = \begin{cases} \text{sgn}(h[-n]), & \text{if } h[-n] \neq 0 \\ K, & \text{if } h[-n] = 0 \end{cases}$$

Stability Condition of an LTI Discrete-Time System

where $\text{sgn}(c) = +1$ if $c > 0$ and $\text{sgn}(c) = -1$ if $c < 0$ and $|K| \le 1$

- Note: Since $|x[n]| \le 1$, $\{x[n]\}$ is obviously bounded
- For this input, $y[n]$ at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} \text{sgn}(h[k])h[k] = S \le B_y < \infty$$

- Therefore, $|y[n]| \le B_y$ implies $S < \infty$

Stability Condition of an LTI Discrete-Time System

- Example - Consider a causal LTI discrete-time system with an impulse response

$$h[n] = (\alpha)^n \mu[n]$$

- For this system

$$S = \sum_{n=-\infty}^{\infty} \left|\alpha^n\right| \mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} \quad \text{if } |\alpha| < 1$$

- Therefore $S < \infty$ if $|\alpha| < 1$ for which the system is BIBO stable
- If $|\alpha| = 1$, the system is not BIBO stable

Causality Condition of an LTI Discrete-Time System

- Let $x_1[n]$ and $x_2[n]$ be two input sequences with

$$x_1[n] = x_2[n] \quad \text{for} \quad n \le n_o$$

- The corresponding output samples at $n = n_o$ of an LTI system with an impulse response $\{h[n]\}$ are then given by

Causality Condition of an LTI Discrete-Time System

$$y_1[n_o] = \sum_{k=-\infty}^{\infty} h[k]x_1[n_o - k] = \sum_{k=0}^{\infty} h[k]x_1[n_o - k]$$

$$+ \sum_{k=-\infty}^{-1} h[k]x_1[n_o - k]$$

$$y_2[n_o] = \sum_{k=-\infty}^{\infty} h[k]x_2[n_o - k] = \sum_{k=0}^{\infty} h[k]x_2[n_o - k]$$

$$+ \sum_{k=-\infty}^{-1} h[k]x_2[n_o - k]$$

Causality Condition of an LTI Discrete-Time System

- If the LTI system is also causal, then

$$y_1[n_o] = y_2[n_o]$$

- As $x_1[n] = x_2[n]$ for $n \le n_o$

$$\sum_{k=0}^{\infty} h[k]x_1[n_o - k] = \sum_{k=0}^{\infty} h[k]x_2[n_o - k]$$

- This implies

$$\sum_{k=-\infty}^{-1} h[k]x_1[n_o - k] = \sum_{k=-\infty}^{-1} h[k]x_2[n_o - k]$$

Causality Condition of an LTI Discrete-Time System

- As $x_1[n] \neq x_2[n]$ for $n > n_o$ the only way the condition

$$\sum_{k=-\infty}^{-1} h[k]x_1[n_o - k] = \sum_{k=-\infty}^{-1} h[k]x_2[n_o - k]$$

will hold if both sums are equal to zero, which is satisfied if

$$h[k] = 0 \quad \text{for } k < 0$$

Causality Condition of an LTI Discrete-Time System

- ⇒ An LTI discrete-time system is **causal** if and only if its impulse response $\{h[n]\}$ is a causal sequence

- <u>Example</u> - The discrete-time system defined by

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

is a causal system as it has a causal impulse response $\{h[n]\} = \{\underset{\uparrow}{\alpha_1} \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\}$

Causality Condition of an LTI Discrete-Time System

- Example - The discrete-time accumulator defined by

$$y[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

is a causal system as it has a causal impulse response given by

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

Causality Condition of an LTI Discrete-Time System

- Example - The factor-of-2 interpolator defined by

$$y[n] = x_u[n] + \tfrac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$$

is noncausal as it has a noncausal impulse response given by

$$\{h[n]\} = \{0.5 \quad \underset{\uparrow}{1} \quad 0.5\}$$

Causality Condition of an LTI Discrete-Time System

- Note: A noncausal LTI discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting an appropriate amount of delay
- For example, a causal version of the factor-of-2 interpolator is obtained by delaying the input by one sample period:

$$y[n] = x_u[n-1] + \frac{1}{2}\left(x_u[n-2] + x_u[n]\right)$$

Finite-Dimensional LTI Discrete-Time Systems

- An important subclass of LTI discrete-time systems is characterized by a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- $x[n]$ and $y[n]$ are, respectively, the input and the output of the system
- $\{d_k\}$ and $\{p_k\}$ are constants characterizing the system

Finite-Dimensional LTI Discrete-Time Systems

- The **order** of the system is given by max(N,M), which is the order of the difference equation
- It is possible to implement an LTI system characterized by a constant coefficient difference equation as here the computation involves two finite sums of products

Finite-Dimensional LTI Discrete-Time Systems

- If we assume the system to be causal, then the output $y[n]$ can be recursively computed using

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=1}^{M} \frac{p_k}{d_0} x[n-k]$$

provided $d_0 \neq 0$

- $y[n]$ can be computed for all $n \geq n_o$, knowing $x[n]$ and the initial conditions

$$y[n_o - 1], y[n_o - 2], \ldots, y[n_o - N]$$

Classification of LTI Discrete-Time Systems

Based on Impulse Response Length -

- If the impulse response $h[n]$ is of finite length, i.e.,

  $$h[n] = 0 \text{ for } n < N_1 \text{ and } n > N_2, \quad N_1 < N_2$$

  then it is known as a **finite impulse response** (FIR) discrete-time system
- The convolution sum description here is

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

Classification of LTI Discrete-Time Systems

- The output $y[n]$ of an FIR LTI discrete-time system can be computed directly from the convolution sum as it is a finite sum of products
- Examples of FIR LTI discrete-time systems are the moving-average system and the linear interpolators

Classification of LTI Discrete-Time Systems

- If the impulse response is of infinite length, then it is known as an **infinite impulse response** (IIR) discrete-time system
- The class of IIR systems we are concerned with in this course are characterized by linear constant coefficient difference equations

Classification of LTI Discrete-Time Systems

- Example - The discrete-time accumulator defined by

$$y[n] = y[n-1] + x[n]$$

is seen to be an IIR system

Classification of LTI Discrete-Time Systems

- Example - The familiar numerical integration formulas that are used to numerically solve integrals of the form

$$y(t) = \int_0^t x(\tau)d\tau$$

can be shown to be characterized by linear constant coefficient difference equations, and hence, are examples of IIR systems

Classification of LTI Discrete-Time Systems

- If we divide the interval of integration into n equal parts of length T, then the previous integral can be rewritten as

$$y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)\,d\tau$$

where we have set $t = nT$ and used the notation

$$y(nT) = \int_{0}^{nT} x(\tau)\,d\tau$$

Classification of LTI Discrete-Time Systems

- Using the trapezoidal method we can write

$$\int_{(n-1)T}^{nT} x(\tau)d\tau = \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

- Hence, a numerical representation of the definite integral is given by

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

Classification of LTI Discrete-Time Systems

- Let $y[n] = y(nT)$ and $x[n] = x(nT)$
- Then

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

reduces to

$$y[n] = y[n-1] + \frac{T}{2}\{x[n] + x[n-1]\}$$

which is recognized as the difference equation representation of a first-order IIR discrete-time system

Classification of LTI Discrete-Time Systems

Based on the Output Calculation Process

- **Nonrecursive System** - Here the output can be calculated sequentially, knowing only the present and past input samples
- **Recursive System** - Here the output computation involves past output samples in addition to the present and past input samples

Classification of LTI Discrete-Time Systems

Based on the Coefficients -

- **Real Discrete-Time System** - The impulse response samples are real valued
- **Complex Discrete-Time System** - The impulse response samples are complex valued

Correlation of Signals

- There are applications where it is necessary to compare one reference signal with one or more signals to determine the similarity between the pair and to determine additional information based on the similarity

Correlation of Signals

- For example, in digital communications, a set of data symbols are represented by a set of unique discrete-time sequences
- If one of these sequences has been transmitted, the receiver has to determine which particular sequence has been received by comparing the received signal with every member of possible sequences from the set

Correlation of Signals

- Similarly, in radar and sonar applications, the received signal reflected from the target is a delayed version of the transmitted signal and by measuring the delay, one can determine the location of the target
- The detection problem gets more complicated in practice, as often the received signal is corrupted by additive ransom noise

Correlation of Signals

Definitions

- A measure of similarity between a pair of energy signals, $x[n]$ and $y[n]$, is given by the cross-correlation sequence $r_{xy}[\ell]$ defined by

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell], \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

- The parameter ℓ called lag, indicates the time-shift between the pair of signals

Correlation of Signals

- $y[n]$ is said to be shifted by ℓ samples to the right with respect to the reference sequence $x[n]$ for positive values of ℓ, and shifted by ℓ samples to the left for negative values of
- The ordering of the subscripts xy in the definition of $r_{xy}[\ell]$ specifies that $x[n]$ is the reference sequence which remains fixed in time while $y[n]$ is being shifted with respect to $x[n]$

Correlation of Signals

- If $y[n]$ is made the reference signal and shift $x[n]$ with respect to $y[n]$, then the corresponding cross-correlation sequence is given by

$$r_{yx}[\ell] = \sum_{n=-\infty}^{\infty} y[n]x[n-\ell]$$

$$= \sum_{m=-\infty}^{\infty} y[m+\ell]x[m] = r_{xy}[-\ell]$$

- Thus, $r_{yx}[\ell]$ is obtained by time-reversing $r_{xy}[\ell]$

Correlation of Signals

- The autocorrelation sequence of $x[n]$ is given by

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n]x[n-\ell]$$

obtained by setting $y[n] = x[n]$ in the definition of the cross-correlation sequence $r_{xy}[\ell]$

- Note: $r_{xx}[0] = \sum_{n=-\infty}^{\infty} x^2[n] = \mathcal{E}_x$, the energy of the signal $x[n]$

Correlation of Signals

- From the relation $r_{yx}[\ell] = r_{xy}[-\ell]$ it follows that $r_{xx}[\ell] = r_{xx}[-\ell]$ implying that $r_{xx}[\ell]$ is an even function for real $x[n]$
- An examination of

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[n-\ell]$$

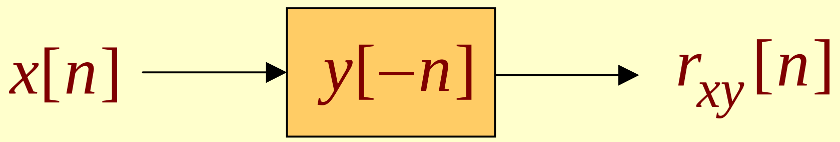
reveals that the expression for the cross-correlation looks quite similar to that of the linear convolution

Correlation of Signals

- This similarity is much clearer if we rewrite the expression for the cross-correlation as

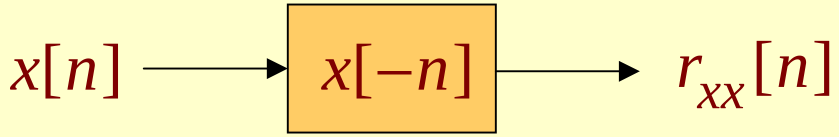
$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n]y[-(\ell - n)] = x[\ell] \circledast y[-\ell]$$

- $\Rightarrow$ The cross-correlation of $y[n]$ with the reference signal $x[n]$ can be computed by processing $x[n]$ with an LTI discrete-time system of impulse response $y[-n]$



Correlation of Signals

- Likewise, the autocorrelation of $x[n]$ can be computed by processing $x[n]$ with an LTI discrete-time system of impulse response $x[-n]$



Properties of Autocorrelation and Cross-correlation Sequences

- Consider two finite-energy sequences x[n] and $y[n]$
- The energy of the combined sequence $a\,x[n]+y[n-\ell]$ is also finite and nonnegative, i.e.,

$$\sum_{n=-\infty}^{\infty}(a\,x[n]+y[n-\ell])^2 = a^2\sum_{n=-\infty}^{\infty}x^2[n]$$

$$+2a\sum_{n=-\infty}^{\infty}x[n]y[n-\ell]+\sum_{n=-\infty}^{\infty}y^2[n-\ell]\geq 0$$

Properties of Autocorrelation and Cross-correlation Sequences

- Thus

$$a^2 r_{xx}[0] + 2a\, r_{xy}[\ell] + r_{yy}[0] \geq 0$$

where $r_{xx}[0] = \mathcal{E}_x > 0$ and $r_{yy}[0] = \mathcal{E}_y > 0$

- We can rewrite the equation on the previous slide as

$$\begin{bmatrix} a & 1 \end{bmatrix} \begin{bmatrix} r_{xx}[0] & r_{xy}[\ell] \\ r_{xy}[\ell] & r_{yy}[0] \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \geq 0$$

for any finite value of a

Properties of Autocorrelation and Cross-correlation Sequences

- Or, in other words, the matrix

$$\begin{bmatrix} r_{xx}[0] & r_{xy}[\ell] \\ r_{xy}[\ell] & r_{yy}[0] \end{bmatrix}$$

is positive semidefinite

- ⇒ $$r_{xx}[0]r_{yy}[0] - r_{xy}^2[\ell] \geq 0$$

or, equivalently,

$$|r_{xy}[\ell]| \leq \sqrt{r_{xx}[0]r_{yy}[0]} = \sqrt{\mathcal{E}_x \mathcal{E}_y}$$

Properties of Autocorrelation and Cross-correlation Sequences

- The last inequality on the previous slide provides an upper bound for the cross-correlation samples
- If we set $y[n] = x[n]$, then the inequality reduces to

$$|r_{xy}[\ell]| \leq r_{xx}[0] = \mathcal{E}_x$$

Properties of Autocorrelation and Cross-correlation Sequences

- Thus, at zero lag ($\ell = 0$), the sample value of the autocorrelation sequence has its maximum value
- Now consider the case

$$y[n] = \pm b\, x[n - N]$$

where N is an integer and $b > 0$ is an arbitrary number

- In this case $\mathcal{E}_y = b^2 \mathcal{E}_x$

Properties of Autocorrelation and Cross-correlation Sequences

- Therefore

$$\sqrt{\mathcal{E}_x \mathcal{E}_y} = \sqrt{b^2 \mathcal{E}_x^2} = b\,\mathcal{E}_x$$

- Using the above result in

$$|r_{xy}[\ell]| \le \sqrt{r_{xx}[0] r_{yy}[0]} = \sqrt{\mathcal{E}_x \mathcal{E}_y}$$

we get

$$-b\,r_{xx}[0] \le r_{xy}[\ell] \le b\,r_{xx}[0]$$

Correlation Computation Using MATLAB

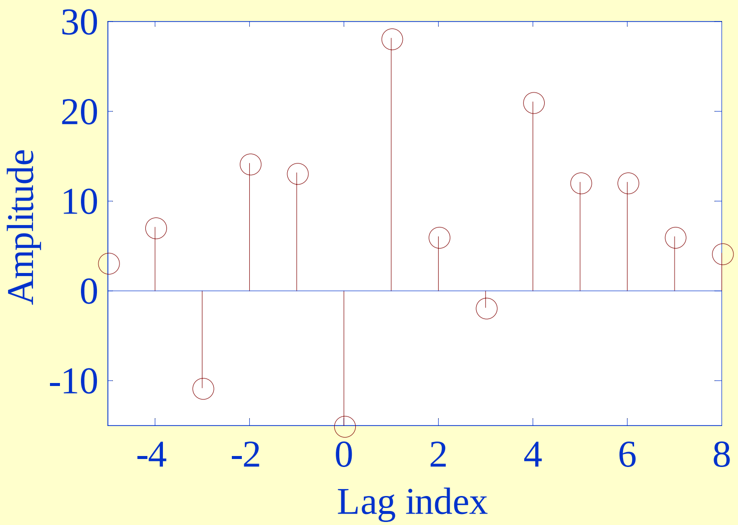
- The cross-correlation and autocorrelation sequences can easily be computed using MATLAB
- Example - Consider the two finite-length sequences

$$x[n] = [1 \;\; 3 \;\; -2 \;\; 1 \;\; 2 \;\; -1 \;\; 4 \;\; 4 \;\; 2]$$

$$y[n] = [2 \;\; -1 \;\; 4 \;\; 1 \;\; -2 \;\; 3]$$

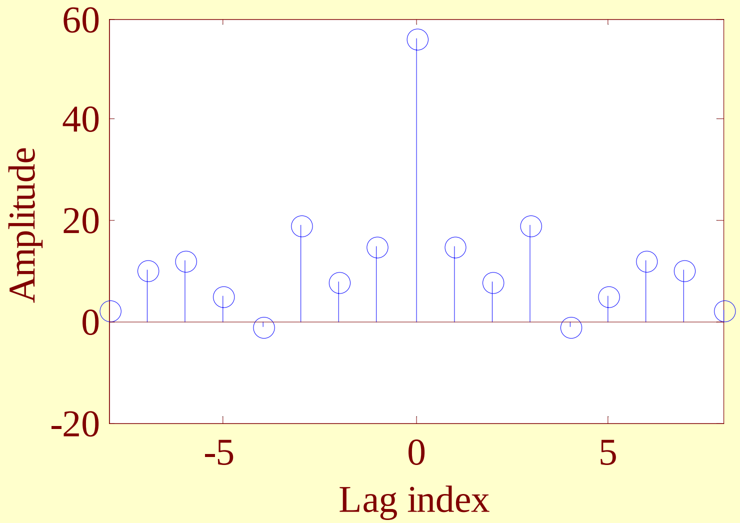
Correlation Computation Using MATLAB

- The cross-correlation sequence $r_{xy}[n]$ computed using Program 2_7 of text is plotted below



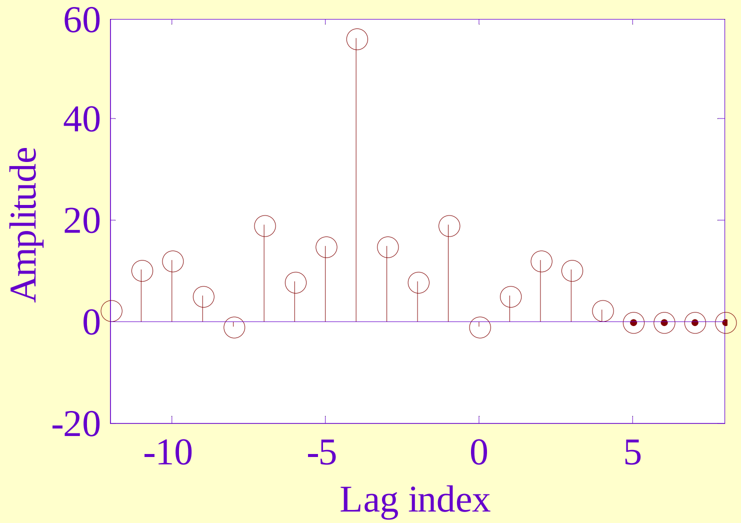
Correlation Computation Using MATLAB

- The autocorrelation sequence $r_{xx}[\ell]$ computed using Program 2_7 is shown below
- Note: At zero lag, $r_{xx}[0]$ is the maximum



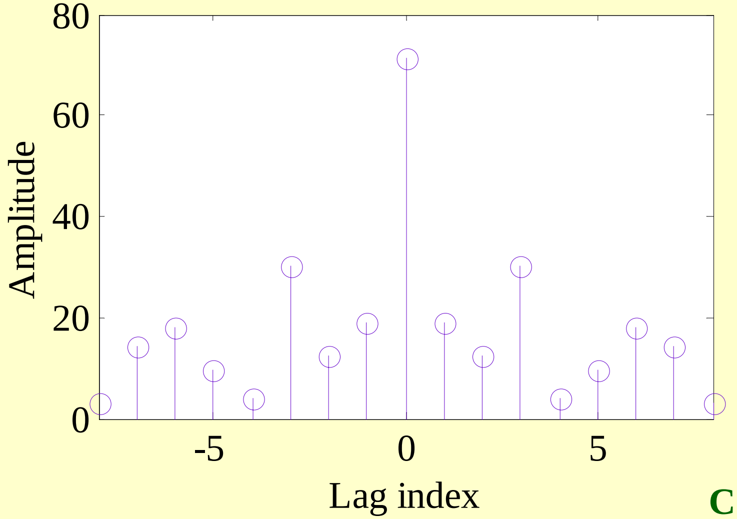
Correlation Computation Using MATLAB

- The plot below shows the cross-correlation of $x[n]$ and $y[n] = x[n-N]$ for $N = 4$
- Note: The peak of the cross-correlation is precisely the value of the delay N



Correlation Computation Using MATLAB

- The plot below shows the autocorrelation of $x[n]$ corrupted with an additive random noise generated using the function `randn`
- Note: The autocorrelation still exhibits a peak at zero lag



Correlation Computation Using MATLAB

- The autocorrelation and the cross-correlation can also be computed using the function `xcorr`
- However, the correlation sequences generated using this function are the time-reversed version of those generated using Programs 2_7 and 2_8

Normalized Forms of Correlation

- Normalized forms of autocorrelation and cross-correlation are given by

$$\rho_{xx}[\ell] = \frac{r_{xx}[\ell]}{r_{xx}[0]}, \qquad \rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_{xx}[0] r_{yy}[0]}}$$

- They are often used for convenience in comparing and displaying
- Note: $|\rho_{xx}[\ell]| \leq 1$ and $|\rho_{xy}[\ell]| \leq 1$ independent of the range of values of $x[n]$ and $y[n]$

Correlation Computation for Power Signals

- The cross-correlation sequence for a pair of power signals, $x[n]$ and $y[n]$, is defined as

$$r_{xy}[\ell] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]y[n-\ell]$$

- The autocorrelation sequence of a power signal $x[n]$ is given by

$$r_{xx}[\ell] = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} x[n]x[n-\ell]$$

Correlation Computation for Periodic Signals

- The cross-correlation sequence for a pair of periodic signals of period N, $\tilde{x}[n]$ and $\tilde{y}[n]$, is defined as

$$r_{\tilde{x}\tilde{y}}[\ell] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{y}[n-\ell]$$

- The autocorrelation sequence of a periodic signal $\tilde{x}[n]$ of period N is given by

$$r_{\tilde{x}\tilde{x}}[\ell] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]\tilde{x}[n-\ell]$$

Correlation Computation for Periodic Signals

- Note: Both $r_{\tilde{x}\tilde{y}}[\ell]$ and $r_{\tilde{x}\tilde{x}}[\ell]$ are also periodic signals with a period N
- The periodicity property of the autocorrelation sequence can be exploited to determine the period of a periodic signal that may have been corrupted by an additive random disturbance

Correlation Computation for Periodic Signals

- Let $\tilde{x}[n]$ be a periodic signal corrupted by the random noise $d[n]$ resulting in the signal

$$w[n] = \tilde{x}[n] + d[n]$$

which is observed for $0 \le n \le M-1$ where $M \gg N$

Correlation Computation for Periodic Signals

- The autocorrelation of $w[n]$ is given by

$$r_{ww}[\ell] = \frac{1}{M}\sum_{n=0}^{M-1} w[n]w[n-\ell]$$

$$= \frac{1}{M}\sum_{n=0}^{M-1} (\tilde{x}[n] + d[n])(\tilde{x}[n-\ell] + d[n-\ell])$$

$$= \frac{1}{M}\sum_{n=0}^{M-1} \tilde{x}[n]\tilde{x}[n-\ell] + \frac{1}{M}\sum_{n=0}^{M-1} d[n]d[n-\ell]$$

$$+ \frac{1}{M}\sum_{n=0}^{M-1} \tilde{x}[n]d[n-\ell] + \frac{1}{M}\sum_{n=0}^{M-1} d[n]\tilde{x}[n-\ell]$$

$$= r_{\tilde{x}\tilde{x}}[\ell] + r_{dd}[\ell] + r_{\tilde{x}d}[\ell] + r_{d\tilde{x}}[\ell]$$

Correlation Computation for Periodic Signals

- In the last equation on the previous slide, $r_{\tilde{x}\tilde{x}}[\ell]$ is a periodic sequence with a period N and hence will have peaks at $\ell = 0, N, 2N, \ldots$ with the same amplitudes as ℓ approaches M
- As $\tilde{x}[n]$ and $d[n]$ are not correlated, samples of cross-correlation sequences $r_{\tilde{x}d}[\ell]$ and $r_{d\tilde{x}}[\ell]$ are likely to be very small relative to the amplitudes of $r_{\tilde{x}\tilde{x}}[\ell]$

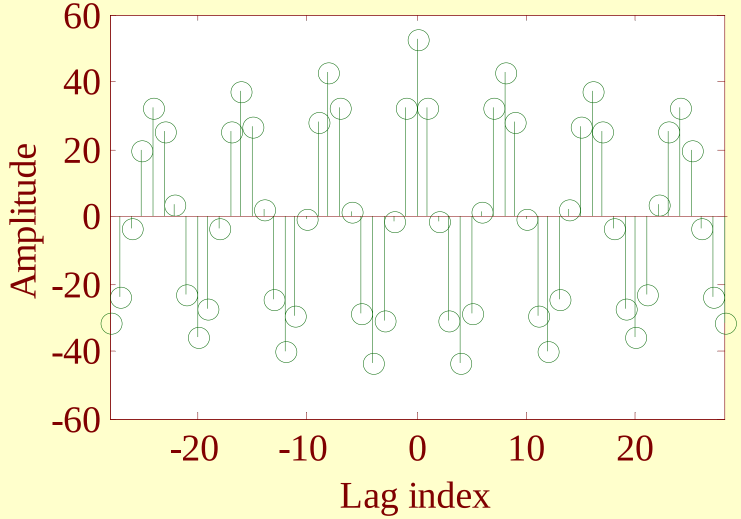
Correlation Computation for Periodic Signals

- The autocorrelation $r_{dd}[\ell]$ of $d[n]$ will show a peak at $\ell = 0$ with other samples having rapidly decreasing amplitudes with increasing values of $|\ell|$
- Hence, peaks of $r_{ww}[\ell]$ for $\ell > 0$ are essentially due to the peaks of $r_{\tilde{x}\tilde{x}}[\ell]$ and can be used to determine whether $\tilde{x}[n]$ is a periodic sequence and also its period N if the peaks occur at periodic intervals

Correlation Computation of a Periodic Signal Using MATLAB

- Example - We determine the period of the sinusoidal sequence $x[n] = \cos(0.25n)$, $0 \le n \le 95$ corrupted by an additive uniformly distributed random noise of amplitude in the range $[-0.5, 0.5]$
- Using Program 2_8 of text we arrive at the plot of $r_{ww}[\ell]$ shown on the next slide

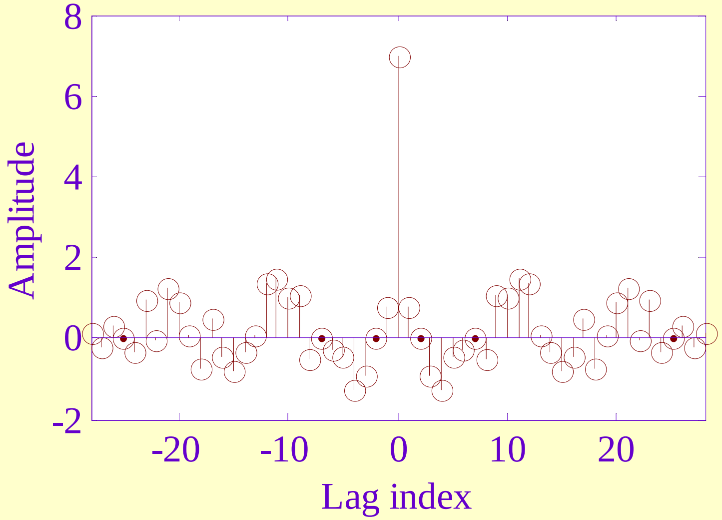
Correlation Computation of a Periodic Signal Using MATLAB



- As can be seen from the plot given above, there is a strong peak at zero lag
- However, there are distinct peaks at lags that are multiples of 8 indicating the period of the sinusoidal sequence to be 8 as expected

Correlation Computation of a Periodic Signal Using MATLAB

- Figure below shows the plot of $r_{dd}[\ell]$



- As can be seen $r_{dd}[\ell]$ shows a very strong peak at only zero lag