

Transform-Domain Representation of Discrete-Time Signals

- Three useful representations of discrete-time sequences in the transform domain:
 - Discrete-time Fourier Transform
 - Discrete Fourier Transform
 - z Transform

Discrete-Time Fourier Transform

- Definition - The discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j X_{im}(e^{j\omega})$$

Discrete-Time Fourier Transform

- $X_{re}(e^{j\omega})$ and $X_{im}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω
- $X(e^{j\omega})$ can alternately be expressed as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$

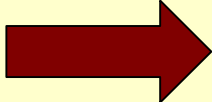
where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}$$

Discrete-Time Fourier Transform

- $|X(e^{j\omega})|$ is called the **magnitude function**
- $\theta(\omega)$ is called the **phase function**
- Both quantities are again real functions of ω
- In many applications, the DTFT is called the **Fourier spectrum**
- Likewise, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called the **magnitude and phase spectra**

Discrete-Time Fourier Transform

- For a real sequence $x[n]$, $|X(e^{j\omega})|$ and $X_{re}(e^{j\omega})$ are even functions of ω , whereas, $\theta(\omega)$ and $X_{im}(e^{j\omega})$ are odd functions of ω
- Note:
$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega+2\pi k)}$$
$$= |X(e^{j\omega})| e^{j\theta(\omega)}$$
for any integer k
-  The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT

Discrete-Time Fourier Transform

- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

called the **principal value**

Discrete-Time Fourier Transform

- The DTFTs of some sequences exhibit discontinuities of 2π in their phase responses
- An alternate type of phase function that is a continuous function of ω is often used
- It is derived from the original phase function by removing the discontinuities of 2π

Discrete-Time Fourier Transform

- The process of removing the discontinuities is called “unwrapping”
- The continuous phase function generated by unwrapping is denoted as $\theta_c(\omega)$
- In some cases, discontinuities of π may be present after unwrapping

Discrete-Time Fourier Transform

- Example - The DTFT of the unit sample sequence $\delta[n]$ is given by

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] = 1$$

- Example - Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

Discrete-Time Fourier Transform

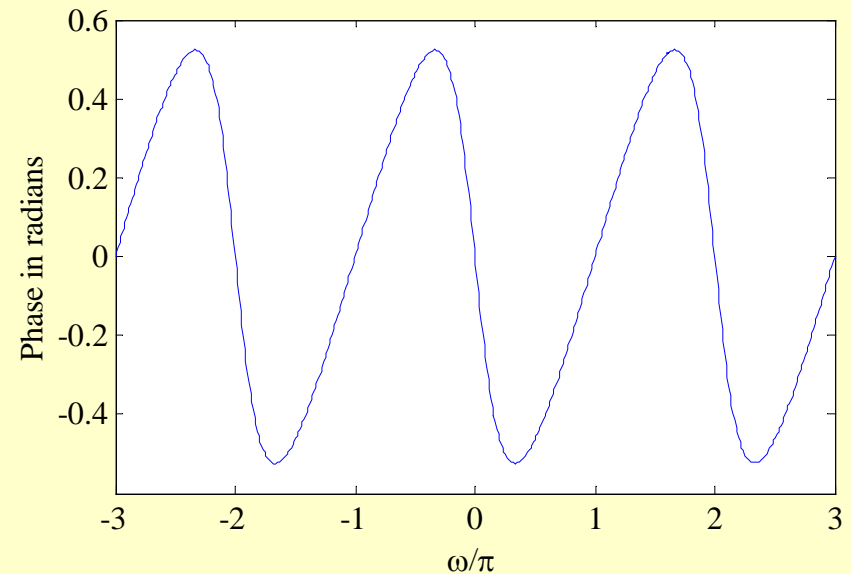
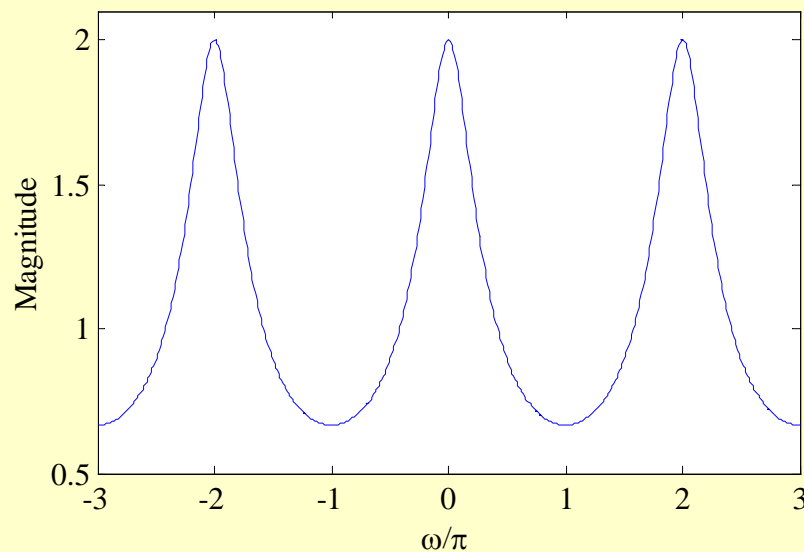
- Its DTFT is given by

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}} \end{aligned}$$

as $\left| \alpha e^{-j\omega} \right| = |\alpha| < 1$

Discrete-Time Fourier Transform

- The magnitude and phase of the DTFT $X(e^{j\omega}) = 1/(1 - 0.5e^{-j\omega})$ are shown below



Discrete-Time Fourier Transform

- The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω
- It is also a periodic function of ω with a period 2π :

$$\begin{aligned} X(e^{j(\omega_o + 2\pi k)}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega_o + 2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_o n} e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_o n} = X(e^{j\omega_o}) \end{aligned}$$

Discrete-Time Fourier Transform

- Therefore

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

represents the Fourier series representation of the periodic function

- As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform

- **Inverse discrete-time Fourier transform:**

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- **Proof:**

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega \ell} \right) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform

- The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists

- **Then**
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell} \right) e^{j\omega n} d\omega$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-\ell)} \right) = \sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n-\ell)}{\pi(n-\ell)}$$

Discrete-Time Fourier Transform

- **Now**
$$\frac{\sin \pi(n - \ell)}{\pi(n - \ell)} = \begin{cases} 1, & n = \ell \\ 0, & n \neq \ell \end{cases}$$
$$= \delta[n - \ell]$$

- **Hence**

$$\sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n - \ell)}{\pi(n - \ell)} = \sum_{\ell=-\infty}^{\infty} x[\ell] \delta[n - \ell] = x[n]$$

Discrete-Time Fourier Transform

- **Convergence Condition** - An infinite series of the form

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

may or may not converge

- Let

$$X_K(e^{j\omega}) = \sum_{n=-K}^K x[n]e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Then for uniform convergence of $X(e^{j\omega})$,

$$\lim_{K \rightarrow \infty} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right| = 0$$

- Now, if $x[n]$ is an absolutely summable sequence, i.e., if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Discrete-Time Fourier Transform

- Then

$$\left| X(e^{j\omega}) \right| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- for all values of ω
- Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X(e^{j\omega})$

Discrete-Time Fourier Transform

- Example - The sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT $X(e^{j\omega})$ therefore converges to $1/(1-\alpha e^{-j\omega})$ uniformly

Discrete-Time Fourier Transform

- Since

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left(\sum_{n=-\infty}^{\infty} |x[n]| \right)^2,$$

an absolutely summable sequence has always a finite energy

- However, a finite-energy sequence is not necessarily absolutely summable

Discrete-Time Fourier Transform

- Example - The sequence

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$E_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

- But, $x[n]$ is not absolutely summable

Discrete-Time Fourier Transform

- To represent a finite energy sequence $x[n]$ that is not absolutely summable by a DTFT $X(e^{j\omega})$, it is necessary to consider a **mean-square convergence** of $X(e^{j\omega})$:

$$\lim_{K \rightarrow \infty} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|^2 d\omega = 0$$

where

$$X_K(e^{j\omega}) = \sum_{n=-K}^K x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Here, the total energy of the error

$$X(e^{j\omega}) - X_K(e^{j\omega})$$

must approach zero at each value of ω as K goes to ∞

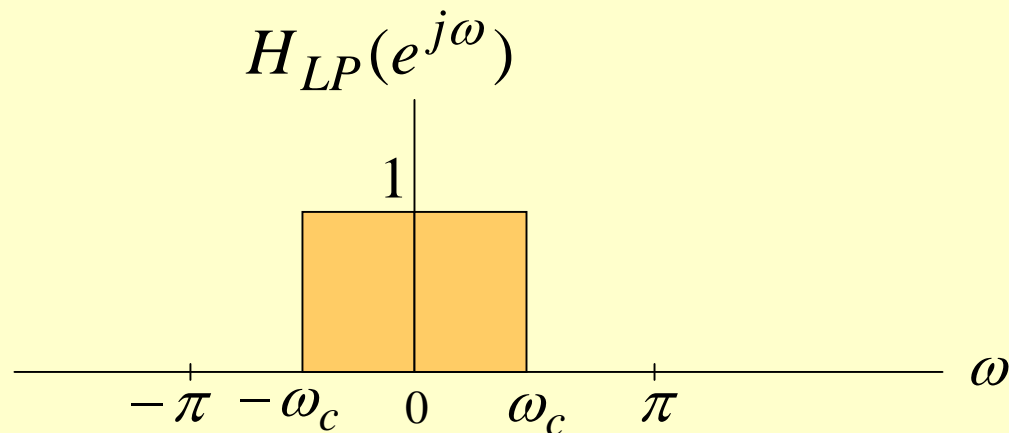
- In such a case, the absolute value of the error $\left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|$ may not go to zero as K goes to ∞ and the DTFT is no longer bounded

Discrete-Time Fourier Transform

- Example - Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$


shown below



Discrete-Time Fourier Transform

- The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$
$$= \frac{1}{2\pi} \left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- The energy of $h_{LP}[n]$ is given by ω_c / π
-  $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable

Discrete-Time Fourier Transform

- As a result

$$\sum_{n=-K}^K h_{LP}[n] e^{-j\omega n} = \sum_{n=-K}^K \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not uniformly converge to $H_{LP}(e^{j\omega})$
for all values of ω , but converges to $H_{LP}(e^{j\omega})$
in the mean-square sense

Discrete-Time Fourier Transform

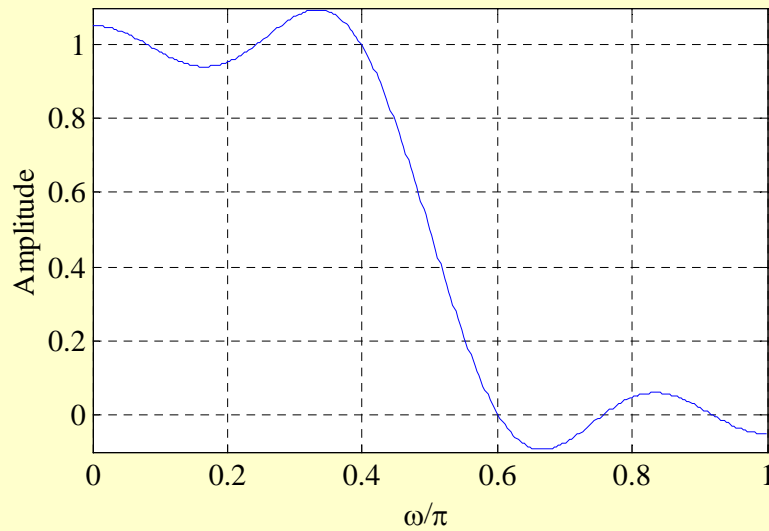
- The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^K \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

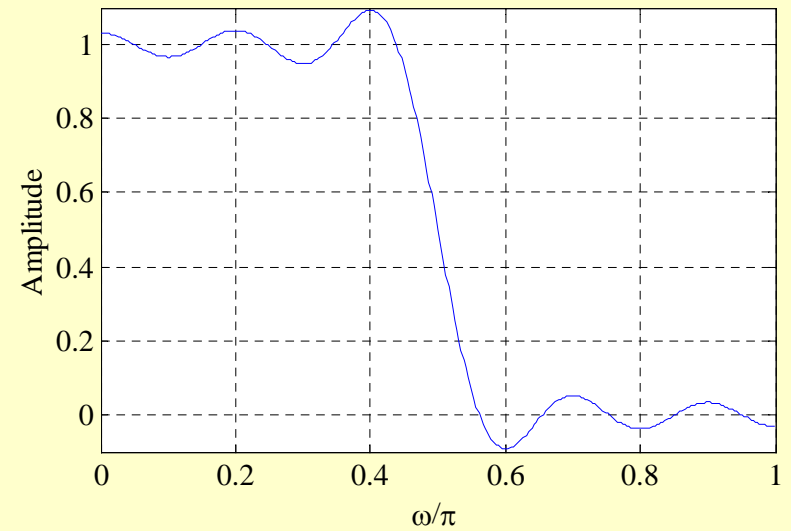
for various values of K as shown next

Discrete-Time Fourier Transform

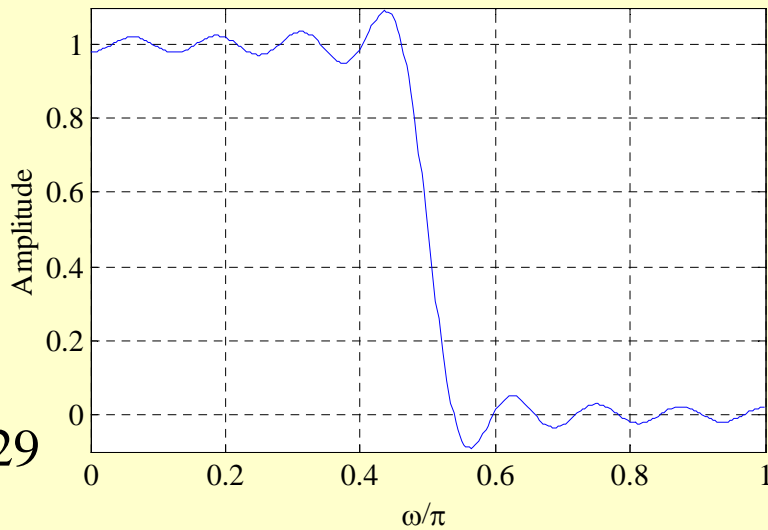
N = 10



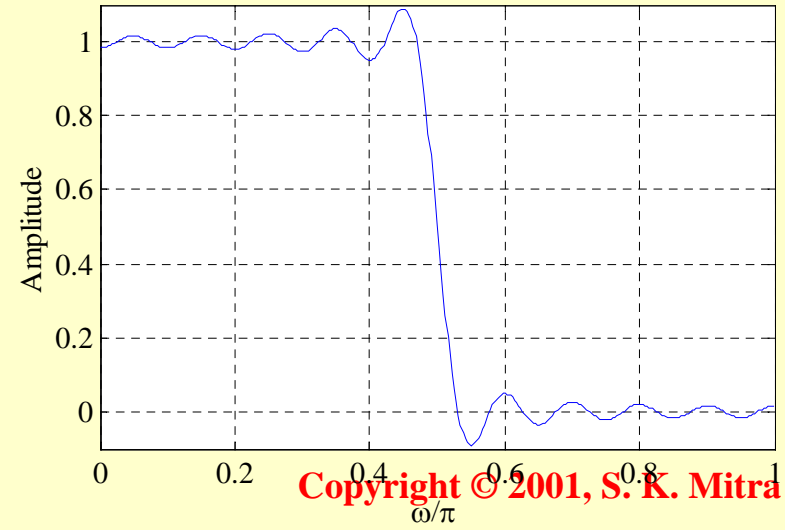
N = 20



N = 30



N = 40



Discrete-Time Fourier Transform

- As can be seen from these plots, independent of the value of K there are ripples in the plot of $H_{LP,K}(e^{j\omega})$ around both sides of the point $\omega = \omega_c$
- The number of ripples increases as K increases with the height of the largest ripple remaining the same for all values of K

Discrete-Time Fourier Transform

- As K goes to infinity, the condition

$$\lim_{K \rightarrow \infty} \int_{-\pi}^{\pi} \left| H_{LP}(e^{j\omega}) - H_{LP,K}(e^{j\omega}) \right|^2 d\omega = 0$$

holds indicating the convergence of $H_{LP,K}(e^{j\omega})$ to $H_{LP}(e^{j\omega})$

- The oscillatory behavior of $H_{LP,K}(e^{j\omega})$ approximating $H_{LP}(e^{j\omega})$ in the mean-square sense at a point of discontinuity is known as the **Gibbs phenomenon**

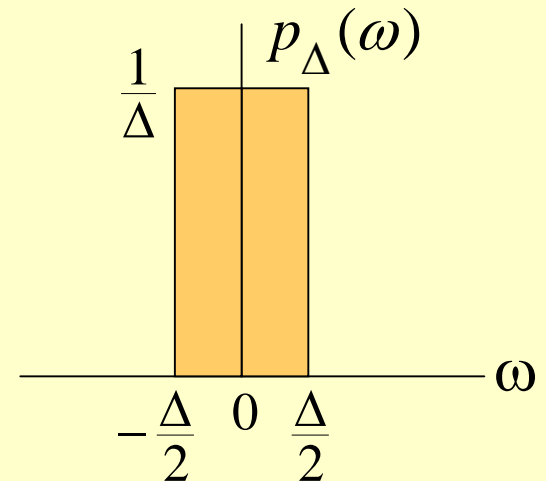
Discrete-Time Fourier Transform

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_0 n + \phi)$ and the exponential sequence $A\alpha^n$
- For this type of sequences, a DTFT representation is possible using the **Dirac delta function** $\delta(\omega)$

Discrete-Time Fourier Transform

- A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function $p_{\Delta}(\omega)$ as Δ goes to zero satisfying

$$\lim_{\Delta \rightarrow 0} \int_{-\infty}^{\infty} p_{\Delta}(\omega) d\omega = \int_{-\infty}^{\infty} \delta(\omega) d\omega$$



Discrete-Time Fourier Transform

- Example - Consider the complex exponential sequence

$$x[n] = e^{j\omega_o n}$$

- Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

where $\delta(\omega)$ is an impulse function of ω and

$$-\pi \leq \omega_o \leq \pi$$

Discrete-Time Fourier Transform

- The function

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

is a periodic function of ω with a period 2π and is called a **periodic impulse train**

- To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_o n}$ we compute the inverse DTFT of $X(e^{j\omega})$

Discrete-Time Fourier Transform

- Thus

$$\begin{aligned}x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_o + 2\pi k) e^{j\omega n} d\omega \\ &= \int_{-\pi}^{\pi} \delta(\omega - \omega_o) e^{j\omega n} d\omega = e^{j\omega_o n}\end{aligned}$$

where we have used the sampling property of the impulse function $\delta(\omega)$

Commonly Used DTFT Pairs

Sequence

DTFT

$$\delta[n] \leftrightarrow 1$$

$$1 \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$$

$$e^{j\omega_0 n} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 + 2\pi k)$$

$$\mu[n] \leftrightarrow \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$

$$\mu[n], (|\alpha| < 1) \leftrightarrow \frac{1}{1 - \alpha e^{-j\omega}}$$

DTFT Properties

- There are a number of important properties of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are quite straightforward
- We illustrate the applications of some of the DTFT properties

Table 3.2: General Properties of DTFT

Type of Property	Sequence	Discrete-Time Fourier Transform
	$g[n]$	$G(e^{j\omega})$
	$h[n]$	$H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_0]$	$e^{-j\omega n_0} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_0 n} g[n]$	$G(e^{j(\omega - \omega_0)})$
Differentiation in frequency	$ng[n]$	$j \frac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \otimes h[n]$	$G(e^{j\omega}) H(e^{j\omega})$
Modulation	$g[n] h[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\theta}) H(e^{j(\omega - \theta)}) d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n] h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega}) H^*(e^{j\omega}) d\omega$	

Table 3.3: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\text{Re}\{x[n]\}$	$X_{\text{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\text{Im}\{x[n]\}$	$X_{\text{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\text{cs}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{ca}}[n]$	$jX_{\text{im}}(e^{j\omega})$

Note: $X_{\text{cs}}(e^{j\omega})$ and $X_{\text{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\text{cs}}[n]$ and $x_{\text{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

Table 3.4: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Symmetry relations	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$ $X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$ $ X(e^{j\omega}) = X(e^{-j\omega}) $ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{\text{ev}}[n]$ and $x_{\text{od}}[n]$ denote the even and odd parts of $x[n]$, respectively.