z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist
- As a result, it is not possible to make use of such frequency-domain characterization in these cases

z-Transform

- A generalization of the DTFT defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

leads to the *z*-transform

- *z*-transform may exist for many sequences for which the DTFT does not exist
- Moreover, use of *z*-transform techniques permits simple algebraic manipulations

z-Transform

- Consequently, z-transform has become an important tool in the analysis and design of digital filters
- For a given sequence $g[n]$, its z-transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

where $z = \mathrm{Re}(z) + j\,\mathrm{Im}(z)$ is a complex variable

z-Transform

- If we let $z = r\,e^{j\omega}$, then the z-transform reduces to

$$G(r\,e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}\,e^{-j\omega n}$$

- The above can be interpreted as the DTFT of the modified sequence $\{g[n]r^{-n}\}$
- For $r = 1$ (i.e., $|z| = 1$), z-transform reduces to its DTFT, provided the latter exists

z-Transform

- The contour $|z| = 1$ is a circle in the z-plane of unity radius and is called the **unit circle**
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- For a given sequence, the set $\mathcal{R}$ of values of z for which its z-transform converges is called the **region of convergence** (ROC)

z-Transform

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} \left| g[n] r^{-n} \right| < \infty$$

z-Transform

- In general, the ROC $\mathcal{R}$ of a z-transform of a sequence $g[n]$ is an annular region of the z-plane:

$$R_{g^-} < |z| < R_{g^+}$$

where $0 \leq R_{g^-} < R_{g^+} \leq \infty$

- Note: The z-transform is a form of a Laurent series and is an analytic function at every point in the ROC

z-Transform

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

- ROC is the annular region $|z| > |\alpha|$

z-Transform

- Example - The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } \left|\alpha z^{-1}\right| < 1$$

by setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } \left|z^{-1}\right| < 1$$

- ROC is the annular region $1 < |z| \leq \infty$

z-Transform

- Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly

- Example - Consider the anti-causal sequence

$$y[n] = -\alpha^n \mu[-n-1]$$

z-Transform

- Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1$$

- ROC is the annular region $|z| < |\alpha|$

z-Transform

- Note: The z-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z-transform is by specifying its ROC

z-Transform

- The DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle
- The existence of the DTFT does not always imply the existence of the z-transform

z-Transform

- Example - The finite energy sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

has a DTFT given by

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

which converges in the mean-square sense

z-Transform

- However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r

- Some commonly used z-transform pairs are listed on the next slide

Table 3.8: Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n \sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Rational z-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

Rational z-Transforms

- The **degree** of the numerator polynomial $P(z)$ is M and the **degree** of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

Rational *z*-Transforms

- A rational *z*-transform can be alternately written in factored form as

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

Rational z-Transforms

- At a root $z = \xi_\ell$ of the numerator polynomial $G(\xi_\ell) = 0$, and as a result, these values of z are known as the **zeros** of $G(z)$
- At a root $z = \lambda_\ell$ of the denominator polynomial $G(\lambda_\ell) \to \infty$, and as a result, these values of z are known as the **poles** of $G(z)$

Rational z-Transforms

- Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

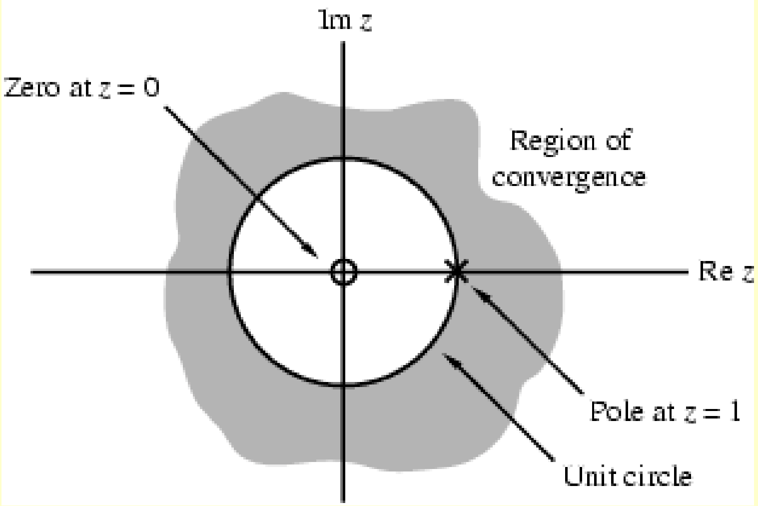
- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z-plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

Rational z-Transforms

- Example - The z-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$

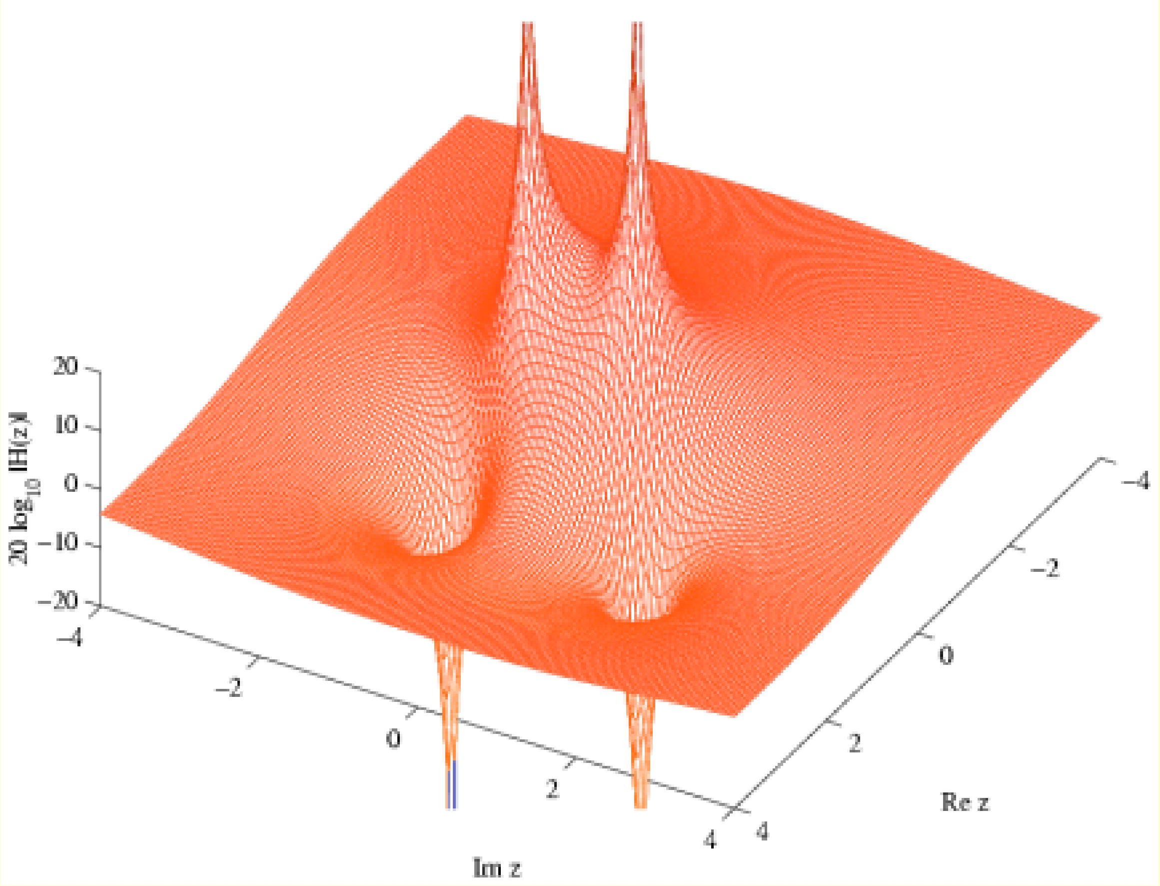


Rational z-Transforms

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

Rational *z*-Transforms



Rational *z*-Transforms

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$

ROC of a Rational z-Transform

- ROC of a z-transform is an important concept
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its z-transform
- Hence, the z-transform must always be specified with its ROC

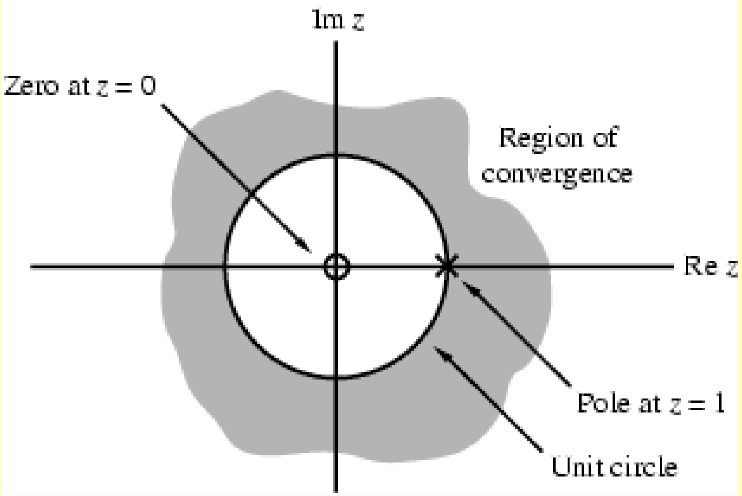
ROC of a Rational *z*-Transform

- Moreover, if the ROC of a z-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z-transform on the unit circle
- There is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

ROC of a Rational z-Transform

- The ROC of a rational z-transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z-transform
- Consider again the pole-zero plot of the z-transform $\mu(z)$

ROC of a Rational z-Transform

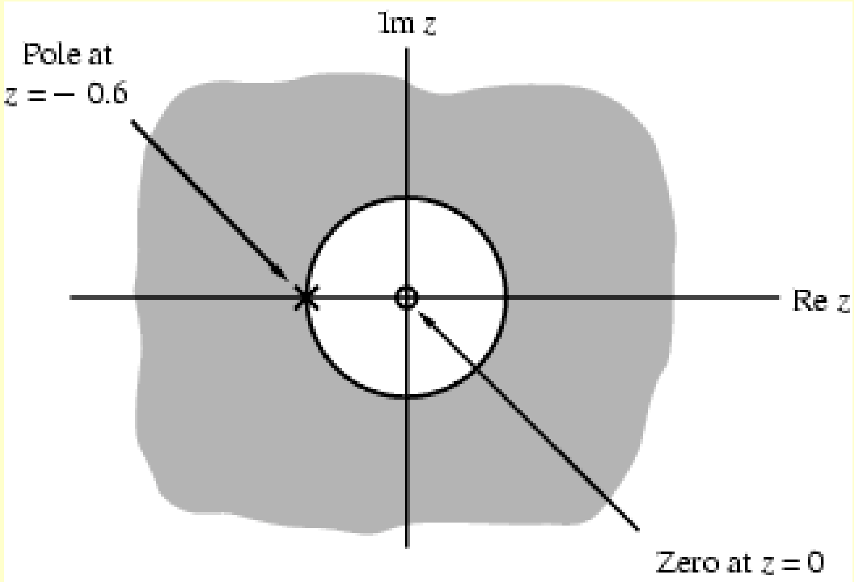


- In this plot, the ROC, shown as the shaded area, is the region of the z-plane just outside the circle centered at the origin and going through the pole at $z = 1$

ROC of a Rational z-Transform

- Example - The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by

$$H(z) = \frac{1}{1 + 0.6\,z^{-1}}, \quad |z| > 0.6$$



- Here the ROC is just outside the circle going through the point $z = -0.6$

ROC of a Rational *z*-Transform

- A sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided
- In general, the ROC depends on the type of the sequence of interest

ROC of a Rational z-Transform

- Example - Consider a finite-length sequence $g[n]$ defined for $-M \leq n \leq N$, where M and N are non-negative integers and $|g[n]| < \infty$
- Its z-transform is given by

$$G(z) = \sum_{n=-M}^{N} g[n] z^{-n} = \frac{\sum_{0}^{N+M} g[n-M] z^{N+M-n}}{z^{N}}$$

ROC of a Rational *z*-Transform

- Note: $G(z)$ has M poles at $z = \infty$ and N poles at $z = 0$
- As can be seen from the expression for $G(z)$, the z-transform of a finite-length bounded sequence converges everywhere in the z-plane except possibly at $z = 0$ and/or at $z = \infty$

ROC of a Rational *z*-Transform

- Example - A **right-sided sequence** with nonzero sample values for $n \geq 0$ is sometimes called a **causal sequence**
- Consider a causal sequence $u_1[n]$
- Its *z*-transform is given by

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$
- On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M nonnegative has a z-transform $U_2(z)$ with M poles at $z = \infty$
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$

ROC of a Rational *z*-Transform

- Example - A **left-sided sequence** with nonzero sample values for $n \leq 0$ is sometimes called a **anticausal sequence**
- Consider an anticausal sequence $v_1[n]$
- Its z-transform is given by

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n}$$

ROC of a Rational z-Transform

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \le N$ with N nonnegative has a z-transform $V_2(z)$ with N poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

ROC of a Rational z-Transform

- Example - The z-transform of a two-sided sequence $w[n]$ can be expressed as

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} w[n] z^{-n} + \sum_{n=-\infty}^{-1} w[n] z^{-n}$$

- The first term on the RHS, $\sum_{n=0}^{\infty} w[n] z^{-n}$, can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$

ROC of a Rational z-Transform

- The second term on the RHS, $\sum_{n=-\infty}^{-1} w[n]z^{-n}$, can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$
- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$
- If $R_5 > R_6$, there is no overlap and the z-transform does not exist

ROC of a Rational *z*-Transform

- Example - Consider the two-sided sequence

$$u[n] = \alpha^n$$

where α can be either real or complex

- Its z-transform is given by

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

- The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$

ROC of a Rational *z*-Transform

- There is no overlap between these two regions
- Hence, the z-transform of $u[n] = \alpha^n$ does not exist

ROC of a Rational z-Transform

- The ROC of a rational z-transform cannot contain any poles and is bounded by the poles
- To show that the z-transform is bounded by the poles, assume that the z-transform $X(z)$ has simple poles at $z = \alpha$ and $z = \beta$
- Assume that the corresponding sequence $x[n]$ is a right-sided sequence

ROC of a Rational z-Transform

- Then $x[n]$ has the form

$$x[n] = \left(r_1\alpha^n + r_2\beta^n\right)\mu[n - N_o], \quad |\alpha| < |\beta|$$

where N_o is a positive or negative integer

- Now, the z-transform of the right-sided sequence $\gamma^n \mu[n - N_o]$ exists if

$$\sum_{n=N_o}^{\infty} \left|\gamma^n z^{-n}\right| < \infty$$

for some z

ROC of a Rational z-Transform

- The condition

$$\sum_{n=N_o}^{\infty} \left|\gamma^n z^{-n}\right| < \infty$$

holds for $|z| > |\gamma|$ but not for $|z| \le |\gamma|$

- Therefore, the *z*-transform of

$$x[n] = \left(r_1\alpha^n + r_2\beta^n\right)\mu[n - N_o], \quad |\alpha| < |\beta|$$

has an ROC defined by $|\beta| < |z| \le \infty$

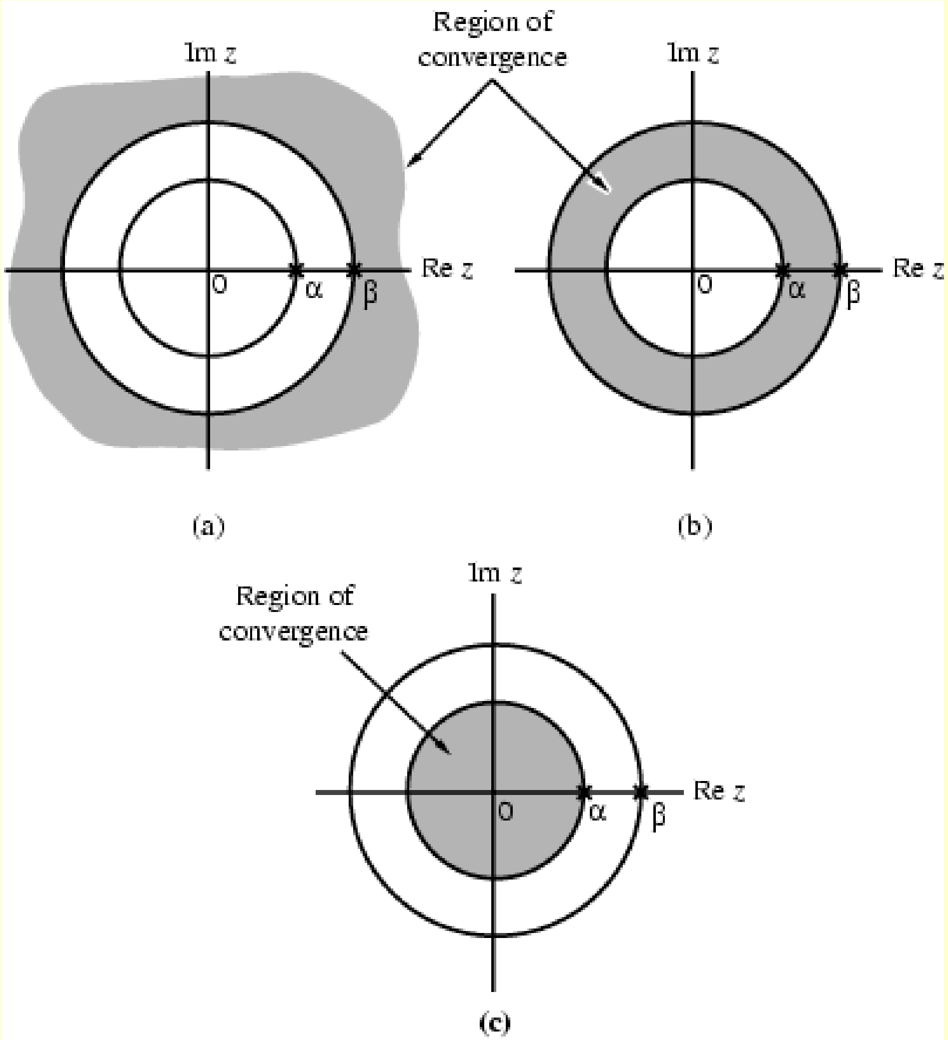
ROC of a Rational z-Transform

- Likewise, the z-transform of a left-sided sequence
$$x[n] = \left(r_1 \alpha^n + r_2 \beta^n\right) \mu[-n - N_o], \quad |\alpha| < |\beta|$$
has an ROC defined by $0 \le |z| < |\alpha|$
- Finally, for a two-sided sequence, some of the poles contribute to terms in the parent sequence for $n < 0$ and the other poles contribute to terms $n \ge 0$

ROC of a Rational z-Transform

- The ROC is thus bounded on the outside by the pole with the smallest magnitude that contributes for $n < 0$ and on the inside by the pole with the largest magnitude that contributes for $n \geq 0$
- There are three possible ROCs of a rational z-transform with poles at $z = \alpha$ and $z = \beta$ $(|\alpha| < |\beta|)$

ROC of a Rational z-Transform



ROC of a Rational *z*-Transform

- In general, if the rational z-transform has N poles with R distinct magnitudes, then it has $R+1$ ROCs
- Thus, there are $R+1$ distinct sequences with the same z-transform
- Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform

ROC of a Rational *z*-Transform

- The ROC of a rational *z*-transform can be easily determined using MATLAB

```
[z,p,k] = tf2zp(num,den)
```

determines the zeros, poles, and the gain constant of a rational *z*-transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`

ROC of a Rational z-Transform

- `[num,den] = zp2tf(z,p,k)` implements the reverse process
- The factored form of the z-transform can be obtained using `sos = zp2sos(z,p,k)`
- The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix `sos`

ROC of a Rational *z*-Transform

$$
sos = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}
$$

where

$$
G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}
$$

ROC of a Rational *z*-Transform

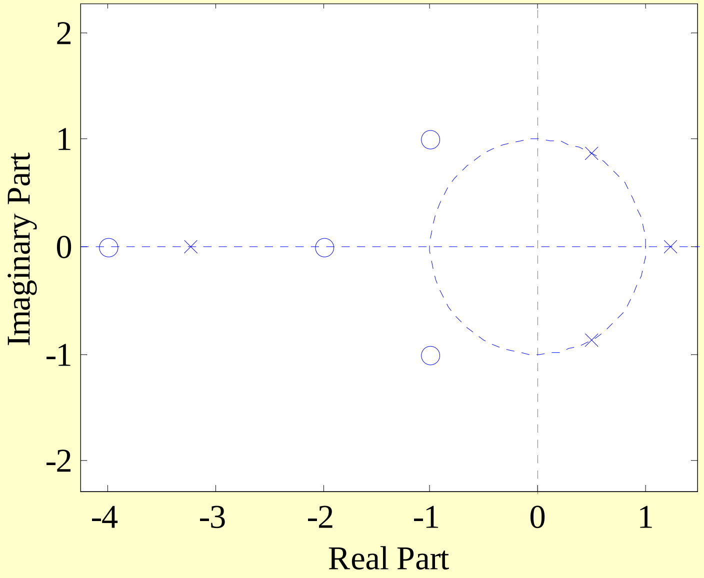
- The pole-zero plot is determined using the function `zplane`
- The *z*-transform can be either described in terms of its `zeros` and `poles`: `zplane(zeros,poles)`
- or, it can be described in terms of its numerator and denominator coefficients: `zplane(num,den)`

ROC of a Rational *z*-Transform

- <u>Example</u> - The pole-zero plot of

$$G(z)=\frac{2z^4+16z^3+44z^2+56z+32}{3z^4+3z^3-15z^2+18z-12}$$

obtained using MATLAB is shown below



$\times -$ pole

$\text{o} -$ zero

Inverse z-Transform

- **General Expression**: Recall that, for $z = r\,e^{j\omega}$, the z-transform $G(z)$ given by

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

is merely the DTFT of the modified sequence $g[n]r^{-n}$

- Accordingly, the inverse DTFT is thus given by

$$g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega}) e^{j\omega n} d\omega$$

Inverse z-Transform

- By making a change of variable $z = r\,e^{j\omega}$, the previous equation can be converted into a contour integral given by

$$g[n] = \frac{1}{2\pi j} \oint_{C'} G(z)\, z^{n-1} dz$$

where C' is a counterclockwise contour of integration defined by $|z| = r$

Inverse z-Transform

- But the integral remains unchanged when is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$
- The contour integral can be evaluated using the Cauchy's residue theorem resulting in

$$g[n] = \sum \begin{bmatrix} \text{residues of } G(z)z^{n-1} \\ \text{at the poles inside } C \end{bmatrix}$$

- The above equation needs to be evaluated at all values of n and is not pursued here

Inverse Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion

Inverse Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N

Inverse Transform by Partial-Fraction Expansion

- The rational function $P_1(z)/D(z)$ is called a **proper fraction**
- Example - Consider

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division we arrive at

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

Inverse Transform by Partial-Fraction Expansion

- **Simple Poles**: In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles
- Let the poles of $G(z)$ be at $z = \lambda_k, 1 \le k \le N$
- A partial-fraction expansion of $G(z)$ is then of the form

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right)$$

Inverse Transform by Partial-Fraction Expansion

- The constants ρ_ℓ in the partial-fraction expansion are called the **residues** and are given by

$$\rho_\ell = (1 - \lambda_\ell z^{-1}) G(z)\Big|_{z=\lambda_\ell}$$

- Each term of the sum in partial-fraction expansion has an ROC given by $|z| > |\lambda_\ell|$ and, thus has an inverse transform of the form $\rho_\ell (\lambda_\ell)^n \mu[n]$

Inverse Transform by Partial-Fraction Expansion

- Therefore, the inverse transform $g[n]$ of $G(z)$ is given by

$$g[n] = \sum_{\ell=1}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]$$

- Note: The above approach with a slight modification can also be used to determine the inverse of a rational z-transform of a noncausal sequence

Inverse Transform by Partial-Fraction Expansion

- Example - Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- A partial-fraction expansion of $H(z)$ is then of the form

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

Inverse Transform by Partial-Fraction Expansion

- Now

$$\rho_1 = (1 - 0.2\,z^{-1})H(z)\big|_{z=0.2} = \left.\frac{1+2\,z^{-1}}{1+0.6\,z^{-1}}\right|_{z=0.2} = 2.75$$

and

$$\rho_2 = (1 + 0.6\,z^{-1})H(z)\big|_{z=-0.6} = \left.\frac{1+2\,z^{-1}}{1-0.2\,z^{-1}}\right|_{z=-0.6} = -1.75$$

Inverse Transform by Partial-Fraction Expansion

- Hence

$$H(z) = \frac{2.75}{1 - 0.2\,z^{-1}} - \frac{1.75}{1 + 0.6\,z^{-1}}$$

- The inverse transform of the above is therefore given by

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n]$$

Inverse Transform by Partial-Fraction Expansion

- **Multiple Poles**: If $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at $z = \nu$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell$, $1 \leq \ell \leq N - L$

Inverse Transform by Partial-Fraction Expansion

- Then the partial-fraction expansion of $G(z)$ is of the form

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1-\lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1-\nu z^{-1})^i}$$

where the constants γ_i are computed using

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1-\nu z^{-1})^L G(z) \right]_{z=\nu},$$

$$1 \le i \le L$$

- The residues ρ_ℓ are calculated as before

Partial-Fraction Expansion Using MATLAB

- `[r,p,k]= residuez(num,den)` develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors `num` and `den`
- Vector `r` contains the residues
- Vector `p` contains the poles
- Vector `k` contains the constants η_ℓ

Partial-Fraction Expansion Using MATLAB

- `[num,den]=residuez(r,p,k)` converts a z-transform expressed in a partial-fraction expansion form to its rational form

Inverse z-Transform via Long Division

- The z-transform $G(z)$ of a causal sequence $\{g[n]\}$ can be expanded in a power series in z^{-1}
- In the series expansion, the coefficient multiplying the term z^{-n} is then the n-th sample $g[n]$
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division

Inverse z-Transform via Long Division

- Example - Consider

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

- Long division of the numerator by the denominator yields

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- As a result

$$\{h[n]\} = \{\underset{\uparrow}{1} \quad 1.6 \quad -0.52 \quad 0.4 \quad -0.2224 \quad \cdots\}, \quad n \geq 0$$

Inverse z-Transform Using MATLAB

- The function `impz` can be used to find the inverse of a rational z-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

Table 3.9: z-Transform Properties

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation		$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$	

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g^-}R_{h^-} < |z| < R_{g^+}R_{h^+}$.

z-Transform Properties

- Example - Consider the two-sided sequence

$$v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1]$$

- Let $x[n] = \alpha^n \mu[n]$ and $y[n] = -\beta^n \mu[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms
- Now $X(z) = \dfrac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$

and $Y(z) = \dfrac{1}{1-\beta z^{-1}}, \quad |z| < |\beta|$

z-Transform Properties

- Using the linearity property we arrive at

$$V(z) = X(z) + Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

- The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$
- If $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$
- If $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist

z-Transform Properties

- Example - Determine the z-transform and its ROC of the causal sequence

$$x[n] = r^n (\cos \omega_o n) \mu[n]$$

- We can express $x[n] = v[n] + v^*[n]$ where

$$v[n] = \tfrac{1}{2} r^n e^{j\omega_o n} \mu[n] = \tfrac{1}{2} \alpha^n \mu[n]$$

- The z-transform of $v[n]$ is given by

$$V(z) = \tfrac{1}{2} \cdot \frac{1}{1 - \alpha z^{-1}} = \tfrac{1}{2} \cdot \frac{1}{1 - r e^{j\omega_o} z^{-1}}, \quad |z| > |\alpha| = r$$

z-Transform Properties

- Using the conjugation property we obtain the *z*-transform of $v^*[n]$ as

$$V^*(z^*) = \frac{1}{2} \cdot \frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2} \cdot \frac{1}{1-r\,e^{-j\omega_o} z^{-1}},$$

$$|z| > |\alpha|$$

- Finally, using the linearity property we get

$$X(z) = V(z) + V^*(z^*)$$

$$= \frac{1}{2}\left(\frac{1}{1-r\,e^{j\omega_o} z^{-1}} + \frac{1}{1-r\,e^{-j\omega_o} z^{-1}}\right)$$

z-Transform Properties

- or,

$$X(z) = \frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}, \quad |z| > r$$

- Example - Determine the *z*-transform $Y(z)$ and the ROC of the sequence

$$y[n] = (n+1)\alpha^n\mu[n]$$

- We can write $y[n] = n\,x[n] + x[n]$ where

$$x[n] = \alpha^n\mu[n]$$

z-Transform Properties

- Now, the z-transform $X(z)$ of $x[n] = \alpha^n \mu[n]$ is given by

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

- Using the differentiation property, we arrive at the z-transform of $n\,x[n]$ as

$$-z\frac{dX(z)}{dz} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \qquad |z| > |\alpha|$$

z-Transform Properties

- Using the linearity property we finally obtain

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

$$= \frac{1}{(1-\alpha z^{-1})^2}, \quad |z| > |\alpha|$$