Comb Filters

• The simple filters discussed so far are characterized either by a single passband and/or a single stopband

• There are applications where filters with multiple passbands and stopbands are required

• The comb filter is an example of such filters
Comb Filters

• In its most general form, a comb filter has a frequency response that is a periodic function of $\omega$ with a period $2\pi/L$, where $L$ is a positive integer.

• If $H(z)$ is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with $L$ delays resulting in a structure with a transfer function given by $G(z) = H(z^L)$. 
Comb Filters

• If $|H(e^{j\omega})|$ exhibits a peak at $\omega_p$, then $|G(e^{j\omega})|$ will exhibit $L$ peaks at $\omega_p k/L$, $0 \leq k \leq L - 1$ in the frequency range $0 \leq \omega < 2\pi$

• Likewise, if $|H(e^{j\omega})|$ has a notch at $\omega_o$, then $|G(e^{j\omega})|$ will have $L$ notches at $\omega_o k/L$, $0 \leq k \leq L - 1$ in the frequency range $0 \leq \omega < 2\pi$

• A comb filter can be generated from either an FIR or an IIR prototype filter
Comb Filters

• For example, the comb filter generated from the prototype lowpass FIR filter \( H_0(z) = \frac{1}{2}(1 + z^{-1}) \) has a transfer function

\[
G_0(z) = H_0(z^L) = \frac{1}{2}(1 + z^{-L})
\]

• \(|G_0(e^{j\omega})|\) has \( L \) notches at \( \omega = (2k+1)\pi/L \) and \( L \) peaks at \( \omega = 2\pi \frac{k}{L} \), \( 0 \leq k \leq L - 1 \), in the frequency range \( 0 \leq \omega < 2\pi \)
Comb Filters

- For example, the comb filter generated from the prototype highpass FIR filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$ has a transfer function

$$G_1(z) = H_1(z^L) = \frac{1}{2}(1 - z^{-L})$$

- $|G_1(e^{j\omega})|$ has $L$ peaks at $\omega = (2k+1)\pi/L$ and $L$ notches at $\omega = 2\pi k/L$, $0 \leq k \leq L-1$, in the frequency range $0 \leq \omega < 2\pi$
Comb Filters

- Depending on applications, comb filters with other types of periodic magnitude responses can be easily generated by appropriately choosing the prototype filter.

- For example, the $M$-point moving average filter

\[ H(z) = \frac{1-z^{-M}}{M(1-z^{-1})} \]

has been used as a prototype.
Comb Filters

• This filter has a peak magnitude at \( \omega = 0 \), and \( M - 1 \) notches at \( \omega = 2\pi\ell / M, 1 \leq \ell \leq M - 1 \).

• The corresponding comb filter has a transfer function

\[
G(z) = \frac{1 - z^{-LM}}{M (1 - z^{-L})}
\]

whose magnitude has \( L \) peaks at \( \omega = 2\pi k / L, 0 \leq k \leq L - 1 \) and \( L(M - 1) \) notches at \( \omega = 2\pi k / LM, 1 \leq k \leq L(M - 1) \).
Allpass Transfer Function

Definition

• An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

$$|A(e^{j\omega})|^2 = 1,$$  

for all $\omega$

is called an allpass transfer function

• An $M$-th order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \ldots + d_1z^{-M+1} + z^{-M}}{1 + d_1z^{-1} + \ldots + d_{M-1}z^{-M+1} + d_Mz^{-M}}$$
Allpass Transfer Function

• If we denote the denominator polynomials of $A_M(z)$ as $D_M(z)$:
  
  $$D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

  then it follows that $A_M(z)$ can be written as:

  $$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

• Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = \frac{1}{r} e^{-j\phi}$
Allpass Transfer Function

• The numerator of a real-coefficient allpass transfer function is said to be the mirror-image polynomial of the denominator, and vice versa.

• We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-$M$ polynomial $D_M(z)$, i.e.,

$$\tilde{D}_M(z) = z^{-M} D_M(z)$$
Allpass Transfer Function

- The expression

\[ A_M(z) = \pm \frac{z^{-M} D(z^{-1})}{D(z)} \]

implies that the poles and zeros of a real-coefficient allpass function exhibit mirror-image symmetry in the \( z \)-plane.

\[ A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}} \]
Allpass Transfer Function

• To show that $|A_M(e^{j\omega})| = 1$ we observe that

$$A_M(z^{-1}) = \pm \frac{z^M D_M(z)}{D_M(z^{-1})}$$

• Therefore

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^M D_M(z)}{D_M(z^{-1})}$$

• Hence

$$|A_M(e^{j\omega})|^2 = A_M(z)A_M(z^{-1})\bigg|_{z=e^{j\omega}} = 1$$
Allpass Transfer Function

• Now, the poles of a causal stable transfer function must lie inside the unit circle in the $z$-plane

• Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle
Allpass Transfer Function

- Figure below shows the principal value of the phase of the 3rd-order allpass function

\[ A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}} \]

- Note the discontinuity by the amount of \(2\pi\) in the phase \(\theta(\omega)\)
Allpass Transfer Function

• If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function $\theta_c(\omega)$ indicated below

• Note: The unwrapped phase function is a continuous function of $\omega$
Allpass Transfer Function

• The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of $\omega$

Properties

• (1) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure
(2) The magnitude function of a stable allpass function $A(z)$ satisfies:

$$
|A(z)| = \begin{cases} 
< 1, & \text{for } |z| > 1 \\
= 1, & \text{for } |z| = 1 \\
> 1, & \text{for } |z| < 1
\end{cases}
$$

(3) Let $\tau(\omega)$ denote the group delay function of an allpass filter $A(z)$, i.e.,

$$
\tau(\omega) = - \frac{d}{d\omega} [\theta_c(\omega)]
$$
Allpass Transfer Function

• The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of $\omega$ so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$

• The group delay of an $M$-th order stable real-coefficient allpass transfer function satisfies:

$$\int_{0}^{\pi} \tau(\omega) d\omega = M\pi$$
A Simple Application

• A simple but often used application of an allpass filter is as a delay equalizer

• Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response

• The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest
• Since \(|A(e^{j\omega})| = 1\), we have
\[ |G(e^{j\omega})A(e^{j\omega})| = |G(e^{j\omega})| \]
• Overall group delay is given by the sum of the group delays of \(G(z)\) and \(A(z)\)
Minimum-Phase and Maximum-Phase Transfer Functions

• Consider the two 1st-order transfer functions:

\[ H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1 \]

• Both transfer functions have a pole inside the unit circle at the same location \( z = -a \) and are stable

• But the zero of \( H_1(z) \) is inside the unit circle at \( z = -b \), whereas, the zero of \( H_2(z) \) is at \( z = -\frac{1}{b} \) situated in a mirror-image symmetry
Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the pole-zero plots of the two transfer functions

\[ H_1(z) \]

\[ H_2(z) \]
Minimum-Phase and Maximum-Phase Transfer Functions

• However, both transfer functions have an identical magnitude function as

\[ H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) = 1 \]

• The corresponding phase functions are

\[ \arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega} \]

\[ \arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega} \]
Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the unwrapped phase responses of the two transfer functions for $a = 0.8$ and $b = -0.5$
Minimum-Phase and Maximum-Phase Transfer Functions

• From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$

• Generalizing the above result, we can show that a causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal transfer function with identical magnitude but having all zeros inside the unit circle
Minimum-Phase and Maximum-Minimum-Phase Transfer Functions

- A causal stable transfer function with all zeros inside the unit circle is called a minimum-phase transfer function.
- A causal stable transfer function with all zeros outside the unit circle is called a maximum-phase transfer function.
- Any nonminimum-phase transfer function can be expressed as the product of a minimum-phase transfer function and a stable allpass transfer function.
Complementary Transfer Functions

• A set of digital transfer functions with complementary characteristics often finds useful applications in practice

• Four useful complementary relations are described next along with some applications
Complementary Transfer Functions

Delay-Complementary Transfer Functions

- A set of $L$ transfer functions, \( \{ H_i(z) \} \), $0 \leq i \leq L−1$, is defined to be delay-complementary of each other if the sum of their transfer functions is equal to some integer multiple of unit delays, i.e.,

\[
\sum_{i=0}^{L-1} H_i(z) = \beta z^{-n_o}, \quad \beta \neq 0
\]

where $n_o$ is a nonnegative integer.
Complementary Transfer Functions

- A delay-complementary pair \( \{H_0(z), H_1(z)\} \) can be readily designed if one of the pairs is a known Type 1 FIR transfer function of odd length.

- Let \( H_0(z) \) be a Type 1 FIR transfer function of length \( M = 2K+1 \).

- Then its delay-complementary transfer function is given by

\[
H_1(z) = z^{-K} - H_0(z)
\]
Complementary Transfer Functions

- Let the magnitude response of $H_0(z)$ be equal to $1 \pm \delta_p$ in the passband and less than or equal to $\delta_s$ in the stopband where $\delta_p$ and $\delta_s$ are very small numbers.

- Now the frequency response of $H_0(z)$ can be expressed as:

$$H_0(e^{j\omega}) = e^{-jK\omega} \tilde{H}_0(\omega)$$

where $\tilde{H}_0(\omega)$ is the amplitude response.
Complementary Transfer Functions

- Its delay-complementary transfer function \( H_1(z) \) has a frequency response given by

\[
H_1(e^{j\omega}) = e^{-jK\omega} \tilde{H}_1(\omega) = e^{-jK\omega}[1 - \tilde{H}_0(\omega)]
\]

- Now, in the passband, \( 1 - \delta_p \leq \tilde{H}_0(\omega) \leq 1 + \delta_p \), and in the stopband, \( -\delta_s \leq \tilde{H}_0(\omega) \leq \delta_s \)

- It follows from the above equation that in the stopband, \( -\delta_p \leq \tilde{H}_1(\omega) \leq \delta_p \) and in the passband, \( 1 - \delta_s \leq \tilde{H}_1(\omega) \leq 1 + \delta_s \)
Complementary Transfer Functions

- As a result, $H_1(z)$ has a complementary magnitude response characteristic to that of $H_0(z)$ with a stopband exactly identical to the passband of $H_0(z)$, and a passband that is exactly identical to the stopband of $H_0(z)$.

- Thus, if $H_0(z)$ is a lowpass filter, $H_1(z)$ will be a highpass filter, and vice versa.
Complementary Transfer Functions

- The frequency $\omega_o$ at which 
  \[ \tilde{H}_0(\omega_o) = \tilde{H}_1(\omega_o) = 0.5 \]
  the gain responses of both filters are 6 dB below their maximum values
- The frequency $\omega_o$ is thus called the 6-dB crossover frequency
Complementary Transfer Functions

• **Example** - Consider the Type 1 bandstop transfer function

\[ H_{BS}(z) = \frac{1}{64} (1 + z^{-2})^4 (1 - 4z^{-2} + 5z^{-4} + 5z^{-8} - 4z^{-10} + z^{-12}) \]

• Its delay-complementary Type 1 bandpass transfer function is given by

\[ H_{BP}(z) = z^{-10} - H_{BS}(z) = \frac{1}{64} (1 - z^{-2})^4 (1 + 4z^{-2} + 5z^{-4} + 5z^{-8} + 4z^{-10} + z^{-12}) \]
Complementary Transfer Functions

- Plots of the magnitude responses of $H_{BS}(z)$ and $H_{BP}(z)$ are shown below.
Complementary Transfer Functions

Allpass Complementary Filters

• A set of $M$ digital transfer functions, $\{H_i(z)\}$, $0 \leq i \leq M - 1$, is defined to be allpass-complementary of each other, if the sum of their transfer functions is equal to an allpass function, i.e.,

$$\sum_{i=0}^{M-1} H_i(z) = A(z)$$
Complementary Transfer Functions

Power-Complementary Transfer Functions

• A set of $M$ digital transfer functions, $\{H_i(z)\}$, $0 \leq i \leq M - 1$, is defined to be power-complementary of each other, if the sum of their square-magnitude responses is equal to a constant $K$ for all values of $\omega$, i.e.,

$$\sum_{i=0}^{M-1} |H_i(e^{j\omega})|^2 = K,$$

for all $\omega$.
Complementary Transfer Functions

• By analytic continuation, the above property is equal to
  \[ \sum_{i=0}^{M-1} H_i(z) H_i(z^{-1}) = K, \quad \text{for all } \omega \]
  for real coefficient \( H_i(z) \)

• Usually, by scaling the transfer functions, the power-complementary property is defined for \( K = 1 \)
Complementary Transfer Functions

• For a pair of power-complementary transfer functions, $H_0(z)$ and $H_1(z)$, the frequency $\omega_o$ where $\left|H_0(e^{j\omega_o})\right|^2 = \left|H_1(e^{j\omega_o})\right|^2 = 0.5$, is called the cross-over frequency.

• At this frequency the gain responses of both filters are 3-dB below their maximum values.

• As a result, $\omega_o$ is called the 3-dB cross-over frequency.
Complementary Transfer Functions

• **Example** - Consider the two transfer functions $H_0(z)$ and $H_1(z)$ given by

$$H_0(z) = \frac{1}{2} [A_0(z) + A_1(z)]$$

$$H_1(z) = \frac{1}{2} [A_0(z) - A_1(z)]$$

where $A_0(z)$ and $A_1(z)$ are stable allpass transfer functions

• **Note that** $H_0(z) + H_1(z) = A_0(z)$

• Hence, $H_0(z)$ and $H_1(z)$ are allpass complementary
Complementary Transfer Functions

• It can be shown that $H_0(z)$ and $H_1(z)$ are also power-complementary

• Moreover, $H_0(z)$ and $H_1(z)$ are bounded-real transfer functions
Doubly-Complementary Transfer Functions

• A set of $M$ transfer functions satisfying both the allpass complementary and the power-complementary properties is known as a doubly-complementary set
Complementary Transfer Functions

- A pair of doubly-complementary IIR transfer functions, $H_0(z)$ and $H_1(z)$, with a sum of allpass decomposition can be simply realized as indicated below.

\[ H_0(z) = \frac{Y_0(z)}{X(z)} \quad \text{and} \quad H_1(z) = \frac{Y_1(z)}{X(z)} \]
Complementary Transfer Functions

- **Example** - The first-order lowpass transfer function

\[
H_{LP}(z) = \frac{1-\alpha}{2} \left( \frac{1+z^{-1}}{1-\alpha z^{-1}} \right)
\]

can be expressed as

\[
H_{LP}(z) = \frac{1}{2} \left( 1 + \frac{-\alpha + z^{-1}}{1-\alpha z^{-1}} \right) = \frac{1}{2} [A_0(z) + A_1(z)]
\]

where

\[
A_0(z) = 1, \quad A_1(z) = \frac{-\alpha + z^{-1}}{1-\alpha z^{-1}}
\]
Complementary Transfer Functions

- Its power-complementary highpass transfer function is thus given by

\[ H_{HP}(z) = \frac{1}{2} [A_0(z) - A_1(z)] = \frac{1}{2} \left( 1 - \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}} \right) \]

\[ = \frac{1 + \alpha}{2} \left( \frac{1 - z^{-1}}{1 - \alpha z^{-1}} \right) \]

- The above expression is precisely the first-order highpass transfer function described earlier
Complementary Transfer Functions

- Figure below demonstrates the allpass complementary property and the power complementary property of $H_{LP}(z)$ and $H_{HP}(z)$.

\[ |H_{HP}(e^{j\omega})| + |H_{LP}(e^{j\omega})| \]
\[ |H_{LP}(e^{j\omega})|^2 + |H_{HP}(e^{j\omega})|^2 \]
Complementary Transfer Functions

Power-Symmetric Filters

• A real-coefficient causal digital filter with a transfer function $H(z)$ is said to be a power-symmetric filter if it satisfies the condition

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K$$

where $K > 0$ is a constant
Complementary Transfer Functions

• It can be shown that the gain function $G(\omega)$ of a power-symmetric transfer function at $\omega = \pi$ is given by

$$10 \log_{10} K - 3 \text{ dB}$$

• If we define $G(z) = H(-z)$, then it follows from the definition of the power-symmetric filter that $H(z)$ and $G(z)$ are power-complementary as

$$H(z)H(z^{-1}) + G(z)G(z^{-1}) = \text{a constant}$$
Conjugate Quadratic Filter

- If a power-symmetric filter has an FIR transfer function $H(z)$ of order $N$, then the FIR digital filter with a transfer function

$$G(z) = z^{-1}H(z^{-1})$$

is called a **conjugate quadratic filter** of $H(z)$ and vice-versa
Complementary Transfer Functions

• It follows from the definition that $G(z)$ is also a power-symmetric causal filter.

• It also can be seen that a pair of conjugate quadratic filters $H(z)$ and $G(z)$ are also power-complementary.
Complementary Transfer Functions

- **Example** - Let \( H(z) = 1 - 2z^{-1} + 6z^{-2} + 3z^{-3} \)

- **We form**

\[
H(z)H(z^{-1}) + H(-z)H(-z^{-1})
= (1 - 2z^{-1} + 6z^{-2} + 3z^{-3})(1 - 2z + 6z^{2} + 3z^{3})
+ (1 + 2z^{-1} + 6z^{-2} - 3z^{-3})(1 + 2z + 6z^{2} - 3z^{3})
= (3z^{3} + 4z + 50 + 4z^{-1} + 3z^{-3})
+ (-3z^{3} - 4z + 50 - 4z^{-1} - 3z^{-3}) = 100
\]

- \( H(z) \) is a power-symmetric transfer function
Digital Two-Pairs

• The LTI discrete-time systems considered so far are single-input, single-output structures characterized by a transfer function

• Often, such a system can be efficiently realized by interconnecting two-input, two-output structures, more commonly called two-pairs
Digital Two-Pairs

• Figures below show two commonly used block diagram representations of a two-pair

• Here $Y_1$ and $Y_2$ denote the two outputs, and $X_1$ and $X_2$ denote the two inputs, where the dependencies on the variable $z$ has been omitted for simplicity
Digital Two-Pairs

• The input-output relation of a digital two-pair is given by

\[
\begin{bmatrix}
Y_1 \\
Y_2
\end{bmatrix} =
\begin{bmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
\]

• In the above relation the matrix \( \tau \) given by

\[
\tau =
\begin{bmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{bmatrix}
\]

is called the \textbf{transfer matrix} of the two-pair
Digital Two-Pairs

It follows from the input-output relation that the transfer parameters can be found as follows:

\[
t_{11} = \left. \frac{Y_1}{X_1} \right|_{X_2=0}, \hspace{1cm} t_{12} = \left. \frac{Y_1}{X_2} \right|_{X_1=0}
\]

\[
t_{21} = \left. \frac{Y_2}{X_1} \right|_{X_2=0}, \hspace{1cm} t_{22} = \left. \frac{Y_2}{X_2} \right|_{X_1=0}
\]
Digital Two-Pairs

• An alternate characterization of the two-pair is in terms of its chain parameters as

\[
\begin{bmatrix}
X_1 \\
Y_1
\end{bmatrix} = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} \begin{bmatrix}
Y_2 \\
X_2
\end{bmatrix}
\]

where the matrix \( \Gamma \) given by

\[
\Gamma = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]

is called the **chain matrix** of the two-pair.
Digital Two-Pairs

• The relation between the transfer parameters and the chain parameters are given by

\[
\begin{align*}
t_{11} &= \frac{C}{A}, \quad t_{12} = \frac{AD - BC}{A}, \quad t_{21} = \frac{1}{A}, \quad t_{22} = -\frac{C}{A} \\
A &= \frac{1}{t_{21}}, \quad B = -\frac{t_{22}}{t_{21}}, \quad C = \frac{t_{11}}{t_{21}}, \quad D = \frac{t_{12}t_{21} - t_{11}t_{22}}{t_{21}}
\end{align*}
\]
Two-Pair Interconnection Schemes

Cascade Connection - $\Gamma$-cascade

\[
\begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix}
\begin{bmatrix}
X_1' \\
Y_1'
\end{bmatrix}
= 
\begin{bmatrix}
A'' & B'' \\
C'' & D''
\end{bmatrix}
\begin{bmatrix}
Y_2' \\
X_2'
\end{bmatrix}
\]

\[
\begin{bmatrix}
X_1'' \\
Y_1''
\end{bmatrix}
= 
\begin{bmatrix}
A'' & B'' \\
C'' & D''
\end{bmatrix}
\begin{bmatrix}
Y_2'' \\
X_2''
\end{bmatrix}
\]

- Here
Two-Pair Interconnection Schemes

• But from figure, \( X''_1 = Y'_2 \) and \( Y''_1 = X'_2 \)

• Substituting the above relations in the first equation on the previous slide and combining the two equations we get

\[
\begin{bmatrix}
X'_1 \\
Y'_1
\end{bmatrix} = \begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix} \begin{bmatrix}
A'' & B'' \\
C'' & D''
\end{bmatrix} \begin{bmatrix}
Y''_2 \\
X''_2
\end{bmatrix}
\]

• Hence,

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix} \begin{bmatrix}
A'' & B'' \\
C'' & D''
\end{bmatrix}
\]
Two-Pair Interconnection Schemes

Cascade Connection - $\tau$-cascade

$$\begin{bmatrix} X'_1 \\ X'_2 \end{bmatrix} \rightarrow \begin{bmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \end{bmatrix} \rightarrow \begin{bmatrix} Y'_1 \\ Y'_2 \end{bmatrix}$$

$$\begin{bmatrix} X''_1 \\ X''_2 \end{bmatrix} \rightarrow \begin{bmatrix} t''_{11} & t''_{12} \\ t''_{21} & t''_{22} \end{bmatrix} \rightarrow \begin{bmatrix} Y''_1 \\ Y''_2 \end{bmatrix}$$

**Here**

$$\begin{bmatrix} Y'_1 \\ Y'_2 \end{bmatrix} = \begin{bmatrix} t'_{11} & t'_{12} \\ t'_{21} & t'_{22} \end{bmatrix} \begin{bmatrix} X'_1 \\ X'_2 \end{bmatrix}$$

$$\begin{bmatrix} Y''_1 \\ Y''_2 \end{bmatrix} = \begin{bmatrix} t''_{11} & t''_{12} \\ t''_{21} & t''_{22} \end{bmatrix} \begin{bmatrix} X''_1 \\ X''_2 \end{bmatrix}$$
Two-Pair Interconnection Schemes

• But from figure, \( X_1'' = Y_1' \) and \( X_2'' = Y_2' \)

• Substituting the above relations in the first equation on the previous slide and combining the two equations we get

\[
\begin{bmatrix}
Y_1'' \\
Y_2''
\end{bmatrix} =
\begin{bmatrix}
t_{11}'' & t_{12}'' \\
t_{21}'' & t_{22}''
\end{bmatrix}
\begin{bmatrix}
t_1' \\
t_2'
\end{bmatrix}
\begin{bmatrix}
t_{11}' & t_{12}' \\
t_{21}' & t_{22}'
\end{bmatrix}
\begin{bmatrix}
X_1' \\
X_2'
\end{bmatrix}
\]

• Hence,

\[
\begin{bmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{bmatrix} =
\begin{bmatrix}
t_{11}'' & t_{12}'' \\
t_{21}'' & t_{22}''
\end{bmatrix}
\begin{bmatrix}
t_{11}' & t_{12}' \\
t_{21}' & t_{22}'
\end{bmatrix}
\]

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Two-Pair Interconnection Schemes

Constrained Two-Pair

- It can be shown that

\[
H(z) = \frac{Y_1}{X_1} = \frac{C + D \cdot G(z)}{A + B \cdot G(z)}
\]

\[
= t_{11} + \frac{t_{12}t_{21}G(z)}{1 - t_{22}G(z)}
\]
Algebraic Stability Test

- We have shown that the BIBO stability of a causal rational transfer function requires that all its poles be inside the unit circle.
- For very high-order transfer functions, it is very difficult to determine the pole locations analytically.
- Root locations can of course be determined on a computer by some type of root finding algorithms.
Algebraic Stability Test

- We now outline a simple algebraic test that does not require the determination of pole locations.

The Stability Triangle

- For a 2nd-order transfer function the stability can be easily checked by examining its denominator coefficients.
Algebraic Stability Test

• Let

\[ D(z) = 1 + d_1 z^{-1} + d_2 z^{-2} \]

denote the denominator of the transfer function

• In terms of its poles, \( D(z) \) can be expressed as

\[ D(z) = (1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) = 1 - (\lambda_1 + \lambda_2) z^{-1} + \lambda_1 \lambda_2 z^{-2} \]

• Comparing the last two equations we get

\[ d_1 = -(\lambda_1 + \lambda_2), \quad d_2 = \lambda_1 \lambda_2 \]
Algebraic Stability Test

• The poles are inside the unit circle if
  \[ |\lambda_1| < 1, \quad |\lambda_2| < 1 \]

• Now the coefficient \( d_2 \) is given by the product of the poles

• Hence we must have
  \[ |d_2| < 1 \]

• It can be shown that the second coefficient condition is given by
  \[ |d_1| < 1 + d_2 \]
Algebraic Stability Test

- The region in the \((d_1, d_2)\)-plane where the two coefficient conditions are satisfied, called the stability triangle, is shown below.

Stability region
• **Example** - Consider the two 2nd-order bandpass transfer functions designed earlier:

\[ H_{BP}'(z) = -0.18819 \frac{1 - z^{-2}}{1 - 0.7343424 z^{-1} + 1.37638 z^{-2}} \]

\[ H_{BP}''(z) = 0.13673 \frac{1 - z^{-2}}{1 - 0.533531 z^{-1} + 0.72654253 z^{-2}} \]
Algebraic Stability Test

- In the case of $H_{BP}(z)$, we observe that
  \[ d_1 = -0.7343424, \quad d_2 = 1.3763819 \]
- Since here $|d_2| > 1$, $H_{BP}(z)$ is unstable
- On the other hand, in the case of $H_{BP}(z)$, we observe that
  \[ d_1 = -0.53353098, \quad d_2 = 0.726542528 \]
- Here, $|d_2| < 1$ and $|d_1| < 1 + d_2$, and hence $H_{BP}(z)$ is BIBO stable
A General Stability Test Procedure

- Let $D_M(z)$ denote the denominator of an $M$-th order causal IIR transfer function $H(z)$:

$$D_M(z) = \sum_{i=0}^{M} d_i z^{-i}$$

where we assume $d_0 = 1$ for simplicity

- Define an $M$-th order allpass transfer function:

$$A_M(z) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$
Algebraic Stability Test

• Or, equivalently

\[ A_M(z) = \frac{d_M + d_{M-1}z^{-1} + d_{M-2}z^{-2} + \cdots + d_1z^{-M+1} + z^{-M}}{1 + d_1z^{-1} + d_2z^{-2} + \cdots + d_{M-1}z^{-M+1} + d_Mz^{-M}} \]

• If we express

\[ D_M(z) = \prod_{i=1}^{M} (1 - \lambda_i z^{-i}) \]

then it follows that

\[ d_M = (-1)^M \prod_{i=1}^{M} \lambda_i \]
Algebraic Stability Test

• Now for stability we must have $|\lambda_i| < 1$, which implies the condition $|d_M| < 1$

• Define

$$k_M = A_M(\infty) = d_M$$

• Then a necessary condition for stability of $A_M(z)$, and hence, the transfer function $H(z)$ is given by

$$k_M^2 < 1$$
Algebraic Stability Test

• Assume the above condition holds

• We now form a new function

\[ A_{M-1}(z) = z \left[ \frac{A_M(z) - k_M}{1 - k_M A_M(z)} \right] = z \left[ \frac{A_M(z) - d_M}{1 - d_M A_M(z)} \right] \]

• Substituting the rational form of \( A_M(z) \) in the above equation we get

\[ A_{M-1}(z) = \frac{d'_{M-1} + d'_{M-2} z^{-1} + \cdots + d'_1 z^{-(M-2)} + z^{-(M-1)}}{1 + d'_1 z^{-1} + \cdots + d'_{M-2} z^{-(M-2)} + d'_{M-1} z^{-(M-1)}} \]
Algebraic Stability Test

where

\[ d'_i = \frac{d_i - d_M d_{M-i}}{1 - d_M^2}, \quad 1 \leq i \leq M - 1 \]

• Hence, \( A_{M-1}(z) \) is an allpass function of order \( M - 1 \)

• Now the poles \( \lambda_o \) of \( A_{M-1}(z) \) are given by the roots of the equation

\[ A_M(\lambda_o) = \frac{1}{k_M} \]
Algebraic Stability Test

• By assumption \( k^2_M < 1 \)
• Hence \(|A_M(\lambda_o)| > 1\)
• If \( A_M(z) \) is a stable allpass function, then

\[
|A_M(z)| \begin{cases} < 1, & \text{for } |z| > 1 \\ = 1, & \text{for } |z| = 1 \\ > 1, & \text{for } |z| < 1 \end{cases}
\]

• Thus, if \( A_M(z) \) is a stable allpass function, then the condition \(|A_M(\lambda_o)| > 1\) holds only if \( |\lambda_o| < 1 \)
Algebraic Stability Test

• Or, in other words $A_{M-1}(z)$ is a stable allpass function

• Thus, if $A_M(z)$ is a stable allpass function and $k_M^2 < 1$, then $A_{M-1}(z)$ is also a stable allpass function of one order lower

• We now prove the converse, i.e., if $A_{M-1}(z)$ is a stable allpass function and $k_M^2 < 1$, then $A_M(z)$ is also a stable allpass function
Algebraic Stability Test

• To this end, we express $A_M(z)$ in terms of $A_{M-1}(z)$ arriving at

$$A_M(z) = \frac{k_M + z^{-1}A_{M-1}(z)}{1 + k_M z^{-1}A_{M-1}(z)}$$

• If $\zeta_o$ is a pole of $A_M(z)$, then

$$\zeta_o^{-1}A_{M-1}(\zeta_o) = -\frac{1}{k_M}$$

• By assumption $k_M^2 < 1$ holds
Algebraic Stability Test

• Therefore, $|\zeta_o^{-1} A_{M-1}(\zeta_o)| > 1$ i.e.,
  $|A_{M-1}(\zeta_o)| > |\zeta_o|

• Assume $A_{M-1}(z)$ is a stable allpass function

• Then $|A_{M-1}(z)| \leq 1$ for $|z| \geq 1$

• Now, if $|\zeta_o| \geq 1$, then because of the above condition $|A_{M-1}(\zeta_o)| \leq 1$

• But the condition $|A_{M-1}(\zeta_o)| > |\zeta_o|$ reduces to $|A_{M-1}(\zeta_o)| > 1$ if $|\zeta_o| \geq 1$
Algebraic Stability Test

• Thus there is a contradiction

• On the other hand, if $|\zeta_o| < 1$ then from $|A_{M-1}(z)| > 1$ for $|z| < 1$

  we have $|A_{M-1}(\zeta_o)| > 1$

• The above condition does not violate the condition $|A_{M-1}(\zeta_o)| > |\zeta_o|$
Algebraic Stability Test

• Thus, if $k_M^2 < 1$ and if $A_{M-1}(z)$ is a stable allpass function, then $A_M(z)$ is also a stable allpass function

• Summarizing, a necessary and sufficient set of conditions for the causal allpass function $A_M(z)$ to be stable is therefore:

  (1) $k_M^2 < 1$ , and
  (2) The allpass function $A_{M-1}(z)$ is stable
Algebraic Stability Test

• Thus, once we have checked the condition $k_M^2 < 1$, we test next for the stability of the lower-order allpass function $A_{M-1}(z)$

• The process is then repeated, generating a set of coefficients:

$$k_M, k_{M-1}, \ldots, k_2, k_1$$

and a set of allpass functions of decreasing order:

$$A_M(z), A_{M-1}(z), \ldots, A_2(z), A_1(z), A_0(z) = 1$$
Algebraic Stability Test

- The allpass function $A_M(z)$ is stable if and only if $k_i^2 < 1$ for $i$

- **Example** - Test the stability of $H(z) = \frac{1}{4z^4 + 3z^3 + 2z^2 + z + 1}$

- From $H(z)$ we generate a 4-th order allpass function

$$A_4(z) = \frac{\frac{1}{4}z^4 + \frac{1}{4}z^3 + \frac{1}{2}z^2 + \frac{3}{4}z + 1}{z^4 + \frac{3}{4}z^3 + \frac{1}{2}z^2 + \frac{1}{4}z + \frac{1}{4}} = \frac{d_4 z^4 + d_3 z^3 + d_2 z^2 + d_1 z + 1}{z^4 + d_1 z^3 + d_2 z^2 + d_3 z + d_4}$$

- **Note:** $k_4 = A_4(\infty) = d_4 = \frac{1}{4} < 1$
Algebraic Stability Test

• Using

\[ d'_i = \frac{d_i - d_4d_{4-i}}{1 - d_4^2}, \quad 1 \leq i \leq 3 \]

we determine the coefficients \( \{d'_i\} \) of the third-order allpass function \( A_3(z) \) from the coefficients \( \{d_i\} \) of \( A_4(z) \):

\[
A_3(z) = \frac{d'_3z^3 + d'_2z^2 + d'_1z + 1}{d'_1z^3 + d'_2z^2 + d'_3z + 1} = \frac{\frac{1}{15}z^3 + \frac{2}{5}z^2 + \frac{11}{15}z + 1}{z^3 + \frac{11}{15}z^2 + \frac{2}{5}z + \frac{1}{15}}
\]
Algebraic Stability Test

• Note: \( k_3 = A_3(\infty) = d'_3 = \frac{1}{15} < 1 \)

• Following the above procedure, we derive the next two lower-order allpass functions:

\[
A_2(z) = \frac{\frac{79}{224} z^2 + \frac{159}{224} z + 1}{z^2 + \frac{159}{224} z + \frac{79}{224}}
\]

\[
A_1(z) = \frac{\frac{53}{101} z + 1}{z + \frac{53}{101}}
\]
Algebraic Stability Test

• Note: \( k_2 = A_2(\infty) = \frac{79}{224} < 1 \)
  \[ k_1 = A_1(\infty) = \frac{53}{101} < 1 \]

• Since all of the stability conditions are satisfied, \( A_4(z) \) and hence \( H(z) \) are stable

• Note: It is not necessary to derive \( A_3(z) \) since \( A_2(z) \) can be tested for stability using the coefficient conditions