Comb Filters

- The simple filters discussed so far are characterized either by a single passband and/or a single stopband
- There are applications where filters with multiple passbands and stopbands are required
- The **comb filter** is an example of such filters

Comb Filters

- In its most general form, a comb filter has a frequency response that is a periodic function of ω with a period $2\pi/L$, where L is a positive integer
- If $H(z)$ is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with L delays resulting in a structure with a transfer function given by $G(z) = H(z^L)$

Comb Filters

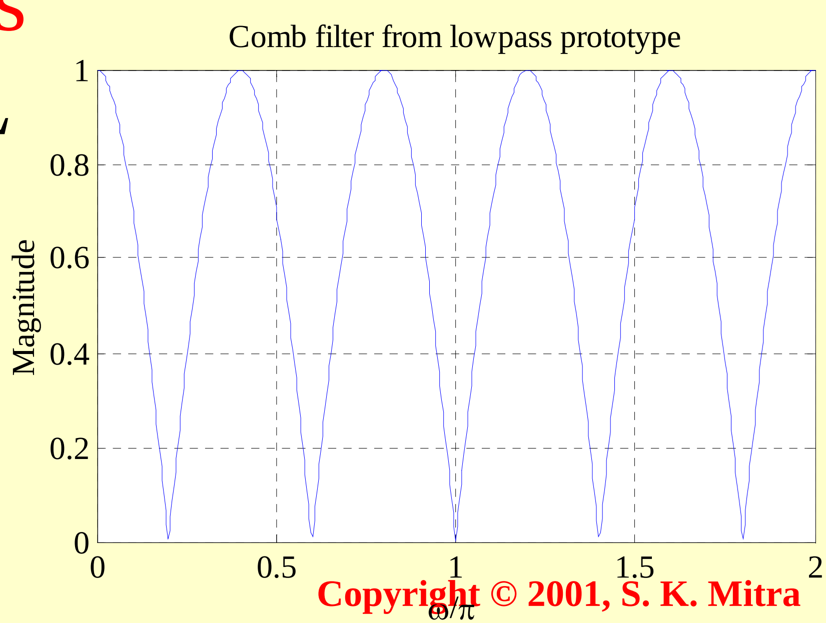
- If $|H(e^{j\omega})|$ exhibits a peak at ω_p, then $|G(e^{j\omega})|$ will exhibit L peaks at $\omega_p k/L$, $0 \leq k \leq L-1$ in the frequency range $0 \leq \omega < 2\pi$
- Likewise, if $|H(e^{j\omega})|$ has a notch at ω_o, then $|G(e^{j\omega})|$ will have L notches at $\omega_o k/L$, $0 \leq k \leq L-1$ in the frequency range $0 \leq \omega < 2\pi$
- A comb filter can be generated from either an FIR or an IIR prototype filter

Comb Filters

- For example, the comb filter generated from the prototype lowpass FIR filter $H_0(z) = \frac{1}{2}(1 + z^{-1})$ has a transfer function

$$G_0(z) = H_0(z^L) = \frac{1}{2}(1 + z^{-L})$$

- $|G_0(e^{j\omega})|$ has L notches at $\omega = (2k+1)\pi/L$ and L peaks at $\omega = 2\pi\, k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$

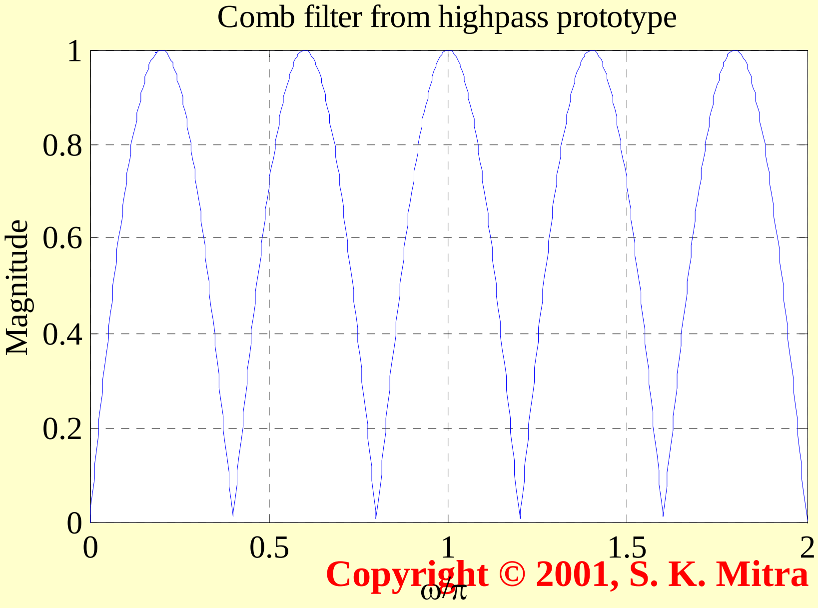


Comb Filters

- For example, the comb filter generated from the prototype highpass FIR filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$ has a transfer function

$$G_1(z) = H_1(z^L) = \tfrac{1}{2}(1 - z^{-L})$$

- $|G_1(e^{j\omega})|$ has L peaks at $\omega = (2k+1)\pi/L$ and L notches at $\omega = 2\pi\, k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$



Comb Filters

- Depending on applications, comb filters with other types of periodic magnitude responses can be easily generated by appropriately choosing the prototype filter
- For example, the M-point moving average filter

$$H(z) = \frac{1 - z^{-M}}{M(1 - z^{-1})}$$

has been used as a prototype

Comb Filters

- This filter has a peak magnitude at $\omega = 0$, and $M-1$ notches at $\omega = 2\pi\ell/M, 1 \le \ell \le M-1$
- The corresponding comb filter has a transfer function

$$G(z) = \frac{1 - z^{-LM}}{M(1 - z^{-L})}$$

whose magnitude has L peaks at $\omega = 2\pi k/L$, $0 \le k \le L-1$ and $L(M-1)$ notches at $\omega = 2\pi k/LM, 1 \le k \le L(M-1)$

Allpass Transfer Function

Definition

- An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

$$|A(e^{j\omega})|^2 = 1, \quad \text{for all } \omega$$

is called an **allpass transfer function**

- An M-th order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

Allpass Transfer Function

- If we denote the denominator polynomials of $A_M(z)$ as $D_M(z)$:

$$D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

then it follows that $A_M(z)$ can be written as:

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = \frac{1}{r} e^{-j\phi}$

Allpass Transfer Function

- The numerator of a real-coefficient allpass transfer function is said to be the **mirror-image polynomial** of the denominator, and vice versa
- We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-M polynomial $D_M(z)$, i.e.,

$$\tilde{D}_M(z) = z^{-M} D_M(z)$$

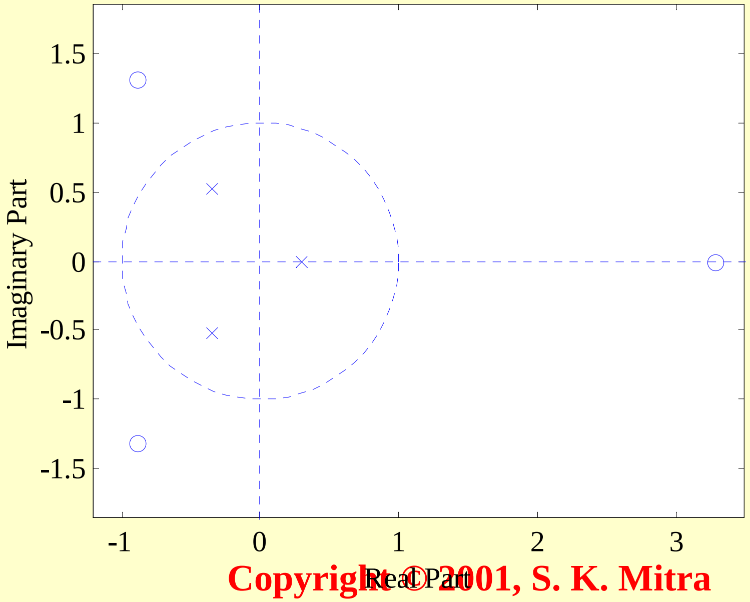
Allpass Transfer Function

- The expression

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

implies that the poles and zeros of a real-coefficient allpass function exhibit **mirror-image symmetry** in the z-plane

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$



Allpass Transfer Function

- To show that $|A_M(e^{j\omega})| = 1$ we observe that

$$A_M(z^{-1}) = \pm \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Therefore

$$A_M(z)A_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^M D_M(z)}{D_M(z^{-1})}$$

- Hence

$$|A_M(e^{j\omega})|^2 = A_M(z)A_M(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

Allpass Transfer Function

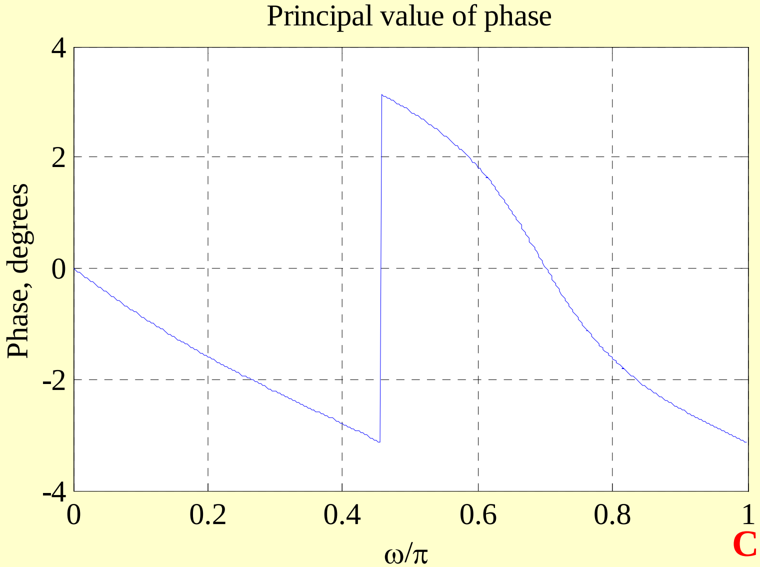
- Now, the poles of a causal stable transfer function must lie inside the unit circle in the z-plane
- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle

Allpass Transfer Function

- Figure below shows the principal value of the phase of the 3rd-order allpass function

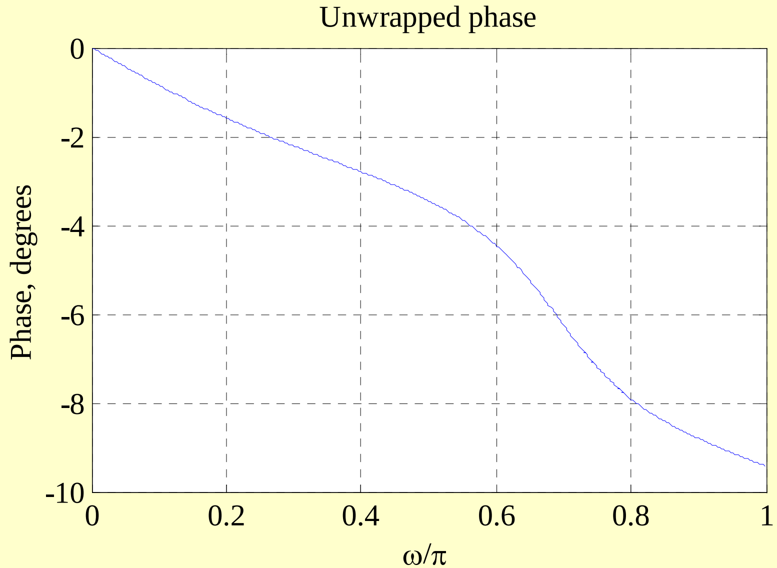
$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

- Note the discontinuity by the amount of 2π in the phase $\theta(\omega)$



Allpass Transfer Function

- If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function $\theta_c(\omega)$ indicated below
- Note: The unwrapped phase function is a continuous function of ω



Allpass Transfer Function

- The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of ω

Properties

- (1) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure

Allpass Transfer Function

- (2) The magnitude function of a stable allpass function $A(z)$ satisfies:

$$|A(z)| \begin{cases} <1, & \text{for } |z|>1 \\ =1, & \text{for } |z|=1 \\ >1, & \text{for } |z|<1 \end{cases}$$

- (3) Let $\tau(\omega)$ denote the group delay function of an allpass filter $A(z)$, i.e.,

$$\tau(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

Allpass Transfer Function

- The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of ω so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$
- The group delay of an M-th order stable real-coefficient allpass transfer function satisfies:

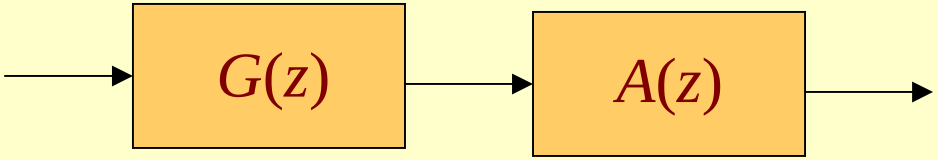
$$\int_0^{\pi} \tau(\omega) d\omega = M\pi$$

Allpass Transfer Function

A Simple Application

- A simple but often used application of an allpass filter is as a **delay equalizer**
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest

Allpass Transfer Function



- Since $|A(e^{j\omega})| = 1$, we have

$$|G(e^{j\omega})A(e^{j\omega})| = |G(e^{j\omega})|$$

- Overall group delay is the given by the sum of the group delays of $G(z)$ and $A(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

- Consider the two 1st-order transfer functions:

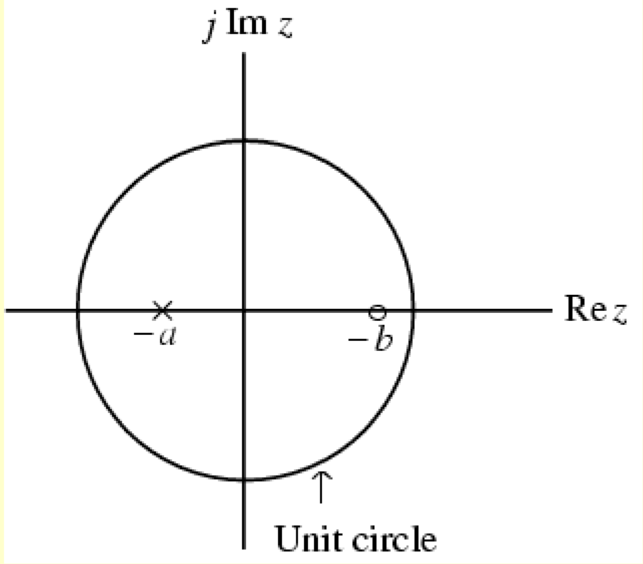
$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1$$

- Both transfer functions have a pole inside the unit circle at the same location $z = -a$ and are stable

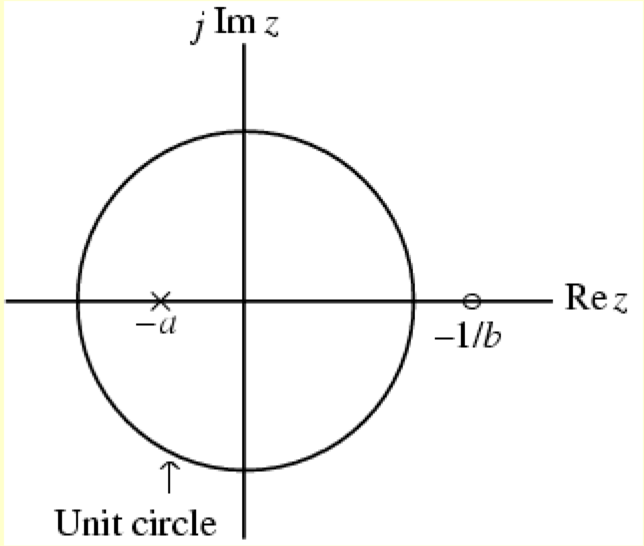
- But the zero of $H_1(z)$ is inside the unit circle at $z = -b$, whereas, the zero of $H_2(z)$ is at $z = -\frac{1}{b}$ situated in a mirror-image symmetry

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the pole-zero plots of the two transfer functions



$H_1(z)$



$H_2(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

- However, both transfer functions have an identical magnitude function as

$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) = 1$$

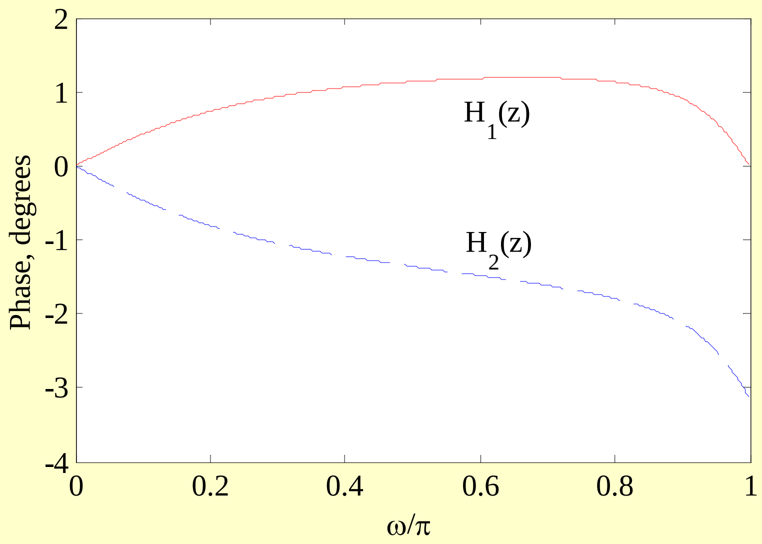
- The corresponding phase functions are

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the unwrapped phase responses of the two transfer functions for $a = 0.8$ and $b = -0.5$



Minimum-Phase and Maximum-Phase Transfer Functions

- From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$
- Generalizing the above result, we can show that a causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal transfer function with identical magnitude but having all zeros inside the unit circle

Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros inside the unit circle is called a **minimum-phase transfer function**
- A causal stable transfer function with all zeros outside the unit circle is called a **maximum-phase transfer function**
- Any nonminimum-phase transfer function can be expressed as the product of a minimum-phase transfer function and a stable allpass transfer function

Complementary Transfer Functions

- A set of digital transfer functions with complementary characteristics often finds useful applications in practice
- Four useful complementary relations are described next along with some applications

Complementary Transfer Functions

Delay-Complementary Transfer Functions

- A set of L transfer functions, $\{H_i(z)\}$, $0 \le i \le L-1$, is defined to be **delay-complementary** of each other if the sum of their transfer functions is equal to some integer multiple of unit delays, i.e.,

$$\sum_{i=0}^{L-1} H_i(z) = \beta z^{-n_o}, \qquad \beta \ne 0$$

where n_o is a nonnegative integer

Complementary Transfer Functions

- A delay-complementary pair $\{H_0(z), H_1(z)\}$ can be readily designed if one of the pairs is a known Type 1 FIR transfer function of odd length
- Let $H_0(z)$ be a Type 1 FIR transfer function of length $M = 2K+1$
- Then its delay-complementary transfer function is given by

$$H_1(z) = z^{-K} - H_0(z)$$

Complementary Transfer Functions

- Let the magnitude response of $H_0(z)$ be equal to $1 \pm \delta_p$ in the passband and less than or equal to δ_s in the stopband where δ_p and δ_s are very small numbers
- Now the frequency response of $H_0(z)$ can be expressed as

$$H_0(e^{j\omega}) = e^{-jK\omega}\tilde{H}_0(\omega)$$

where $\tilde{H}_0(\omega)$ is the **amplitude response**

Complementary Transfer Functions

- Its delay-complementary transfer function $H_1(z)$ has a frequency response given by

$$H_1(e^{j\omega}) = e^{-jK\omega}\tilde{H}_1(\omega) = e^{-jK\omega}[1-\tilde{H}_0(\omega)]$$

- Now, in the passband, $1-\delta_p \le \tilde{H}_0(\omega) \le 1+\delta_p$, and in the stopband, $-\delta_s \le \tilde{H}_0(\omega) \le \delta_s$
- It follows from the above equation that in the stopband, $-\delta_p \le \tilde{H}_1(\omega) \le \delta_p$ and in the passband, $1-\delta_s \le \tilde{H}_1(\omega) \le 1+\delta_s$

Complementary Transfer Functions

- As a result, $H_1(z)$ has a complementary magnitude response characteristic to that of $H_0(z)$ with a stopband exactly identical to the passband of $H_0(z)$, and a passband that is exactly identical to the stopband of $H_0(z)$
- Thus, if $H_0(z)$ is a lowpass filter, $H_1(z)$ will be a highpass filter, and vice versa

Complementary Transfer Functions

- The frequency ω_o at which
$$\tilde{H}_0(\omega_o) = \tilde{H}_1(\omega_o) = 0.5$$
the gain responses of both filters are 6 dB below their maximum values
- The frequency ω_o is thus called the 6-**dB crossover frequency**

Complementary Transfer Functions

- Example - Consider the Type 1 bandstop transfer function

$$H_{BS}(z) = \frac{1}{64}(1+z^{-2})^4(1-4z^{-2}+5z^{-4}+5z^{-8}-4z^{-10}+z^{-12})$$

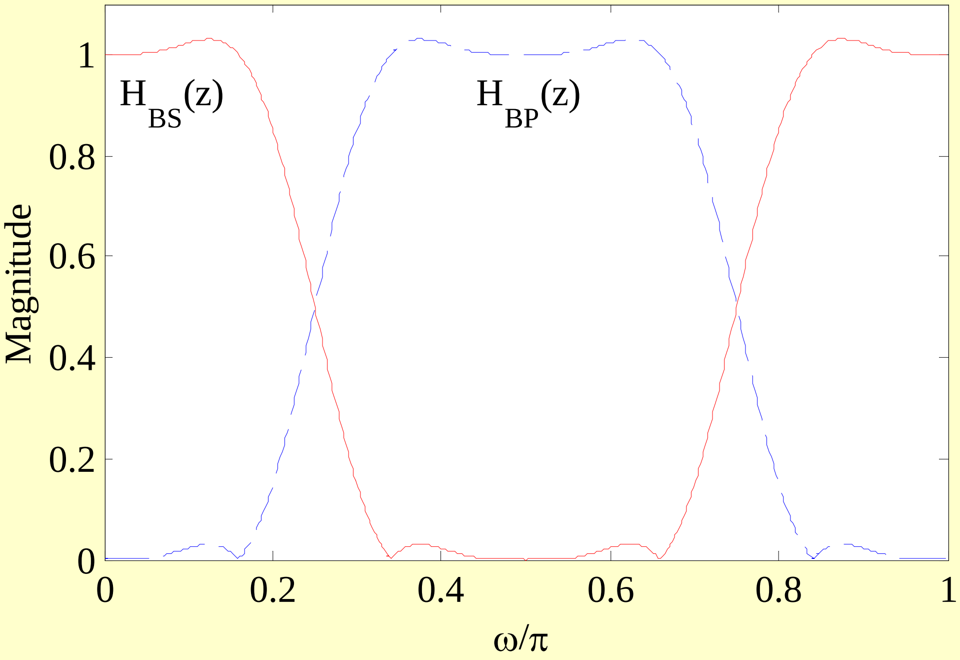
- Its delay-complementary Type 1 bandpass transfer function is given by

$$H_{BP}(z) = z^{-10} - H_{BS}(z)$$

$$= \frac{1}{64}(1-z^{-2})^4(1+4z^{-2}+5z^{-4}+5z^{-8}+4z^{-10}+z^{-12})$$

Complementary Transfer Functions

- Plots of the magnitude responses of $H_{BS}(z)$ and $H_{BP}(z)$ are shown below



Complementary Transfer Functions

Allpass Complementary Filters

- A set of M digital transfer functions, $\{H_i(z)\}$, $0 \le i \le M-1$, is defined to be **allpass-complementary** of each other, if the sum of their transfer functions is equal to an allpass function, i.e.,

$$\sum_{i=0}^{M-1} H_i(z) = A(z)$$

Complementary Transfer Functions

Power-Complementary Transfer Functions

- A set of M digital transfer functions, $\{H_i(z)\}$, $0 \le i \le M-1$, is defined to be **power-complementary** of each other, if the sum of their square-magnitude responses is equal to a constant K for all values of ω, i.e.,

$$\sum_{i=0}^{M-1} \left|H_i(e^{j\omega})\right|^2 = K, \qquad \text{for all } \omega$$

Complementary Transfer Functions

- By analytic continuation, the above property is equal to

$$\sum_{i=0}^{M-1} H_i(z) H_i(z^{-1}) = K, \qquad \text{for all } \omega$$

  for real coefficient $H_i(z)$

- Usually, by scaling the transfer functions, the power-complementary property is defined for $K = 1$

Complementary Transfer Functions

- For a pair of power-complementary transfer functions, $H_0(z)$ and $H_1(z)$, the frequency ω_o where $|H_0(e^{j\omega_o})|^2 = |H_1(e^{j\omega_o})|^2 = 0.5$, is called the **cross-over frequency**
- At this frequency the gain responses of both filters are 3-dB below their maximum values
- As a result, ω_o is called the **3-dB cross-over frequency**

Complementary Transfer Functions

- Example - Consider the two transfer functions $H_0(z)$ and $H_1(z)$ given by

$$H_0(z) = \tfrac{1}{2}[A_0(z) + A_1(z)]$$

$$H_1(z) = \tfrac{1}{2}[A_0(z) - A_1(z)]$$

where $A_0(z)$ and $A_1(z)$ are stable allpass transfer functions

- Note that $H_0(z) + H_1(z) = A_0(z)$
- Hence, $H_0(z)$ and $H_1(z)$ are allpass complementary

Complementary Transfer Functions

- It can be shown that $H_0(z)$ and $H_1(z)$ are also power-complementary
- Moreover, $H_0(z)$ and $H_1(z)$ are bounded-real transfer functions

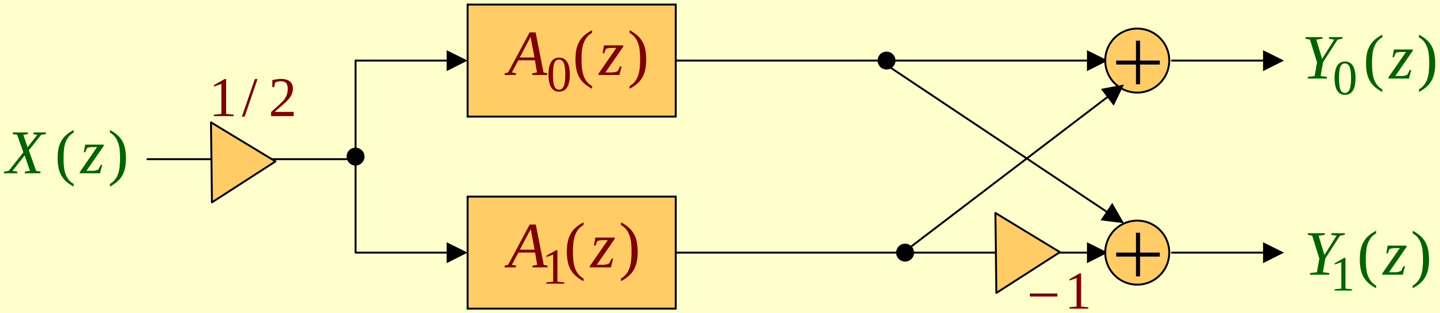
Complementary Transfer Functions

Doubly-Complementary Transfer Functions

- A set of M transfer functions satisfying both the allpass complementary and the power-complementary properties is known as a **doubly-complementary** set

Complementary Transfer Functions

- A pair of doubly-complementary IIR transfer functions, $H_0(z)$ and $H_1(z)$, with a sum of allpass decomposition can be simply realized as indicated below



$$H_0(z) = \frac{Y_0(z)}{X(z)} \qquad H_1(z) = \frac{Y_1(z)}{X(z)}$$

Complementary Transfer Functions

- Example - The first-order lowpass transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

can be expressed as

$$H_{LP}(z) = \frac{1}{2}\left(1+\frac{-\alpha+z^{-1}}{1-\alpha z^{-1}}\right) = \frac{1}{2}[A_0(z) + A_1(z)]$$

where

$$A_0(z) = 1, \qquad A_1(z) = \frac{-\alpha + z^{-1}}{1-\alpha z^{-1}}$$

Complementary Transfer Functions

- Its power-complementary highpass transfer function is thus given by

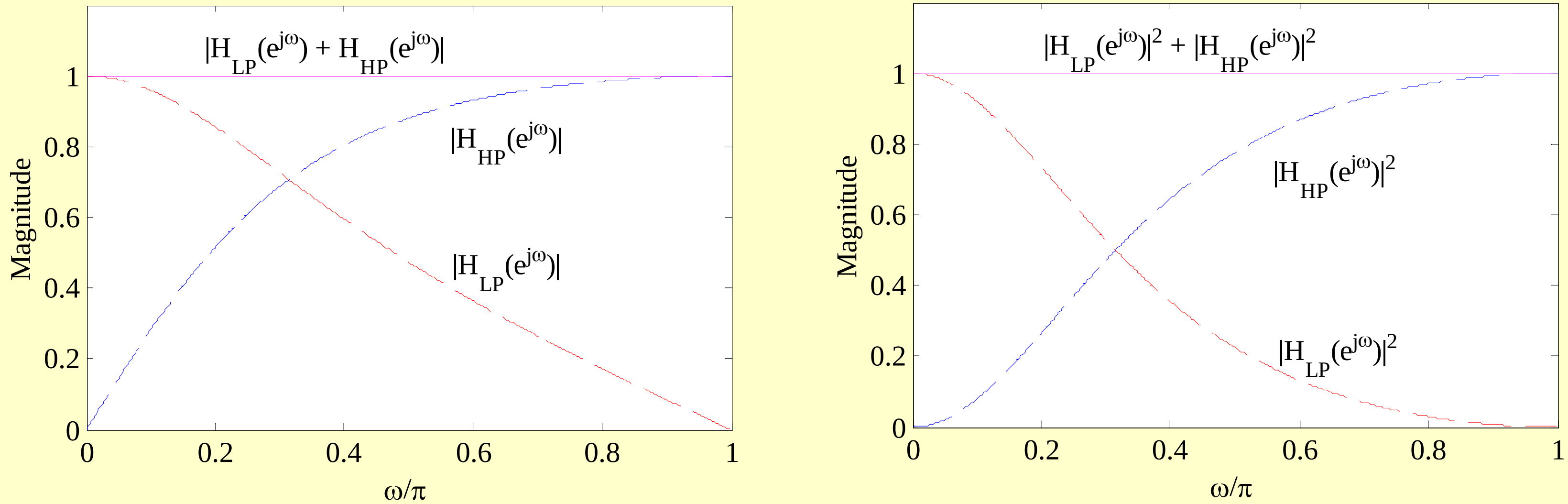
$$H_{HP}(z) = \tfrac{1}{2}[A_0(z) - A_1(z)] = \frac{1}{2}\left(1 - \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}\right)$$

$$= \frac{1+\alpha}{2}\left(\frac{1 - z^{-1}}{1 - \alpha z^{-1}}\right)$$

- The above expression is precisely the first-order highpass transfer function described earlier

Complementary Transfer Functions

- Figure below demonstrates the allpass complementary property and the power complementary property of $H_{LP}(z)$ and $H_{HP}(z)$



Complementary Transfer Functions

Power-Symmetric Filters

- A real-coefficient causal digital filter with a transfer function $H(z)$ is said to be a **power-symmetric filter** if it satisfies the condition

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K$$

where $K > 0$ is a constant

Complementary Transfer Functions

- It can be shown that the gain function $\mathcal{G}(\omega)$ of a power-symmetric transfer function at $\omega = \pi$ is given by

$$10\log_{10} K - 3\ dB$$

- If we define $G(z) = H(-z)$, then it follows from the definition of the power-symmetric filter that $H(z)$ and $G(z)$ are power-complementary as

$$H(z)H(z^{-1}) + G(z)G(z^{-1}) = \text{a constant}$$

Complementary Transfer Functions

Conjugate Quadratic Filter

- If a power-symmetric filter has an FIR transfer function $H(z)$ of order N, then the FIR digital filter with a transfer function

$$G(z) = z^{-1} H(z^{-1})$$

is called a **conjugate quadratic filter** of $H(z)$ and vice-versa

Complementary Transfer Functions

- It follows from the definition that $G(z)$ is also a power-symmetric causal filter
- It also can be seen that a pair of conjugate quadratic filters $H(z)$ and $G(z)$ are also power-complementary

Complementary Transfer Functions

- Example - Let $H(z) = 1 - 2z^{-1} + 6z^{-2} + 3z^{-3}$
- We form

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1})$$

$$= (1 - 2z^{-1} + 6z^{-2} + 3z^{-3})(1 - 2z + 6z^{2} + 3z^{3})$$

$$+ (1 + 2z^{-1} + 6z^{-2} - 3z^{-3})(1 + 2z + 6z^{2} - 3z^{3})$$

$$= (3z^{3} + 4z + 50 + 4z^{-1} + 3z^{-3})$$

$$+ (-3z^{3} - 4z + 50 - 4z^{-1} - 3z^{-3}) = 100$$

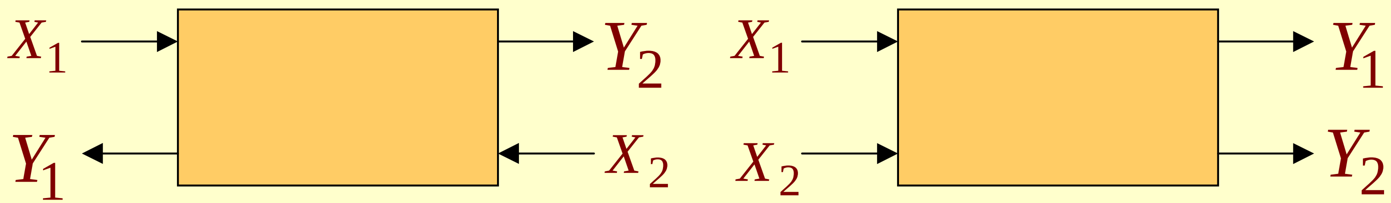
- $H(z)$ is a power-symmetric transfer function

Digital Two-Pairs

- The LTI discrete-time systems considered so far are single-input, single-output structures characterized by a transfer function
- Often, such a system can be efficiently realized by interconnecting two-input, two-output structures, more commonly called **two-pairs**

Digital Two-Pairs

- Figures below show two commonly used block diagram representations of a two-pair



- Here Y_1 and Y_2 denote the two outputs, and X_1 and X_2 denote the two inputs, where the dependencies on the variable z has been omitted for simplicity

Digital Two-Pairs

- The input-output relation of a digital two-pair is given by

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

- In the above relation the matrix τ given by

$$\tau = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

is called the **transfer matrix** of the two-pair

Digital Two-Pairs

- It follows from the input-output relation that the transfer parameters can be found as follows:

$$t_{11} = \left.\frac{Y_1}{X_1}\right|_{X_2=0}, \qquad t_{12} = \left.\frac{Y_1}{X_2}\right|_{X_1=0}$$

$$t_{21} = \left.\frac{Y_2}{X_1}\right|_{X_2=0}, \qquad t_{22} = \left.\frac{Y_2}{X_2}\right|_{X_1=0}$$

Digital Two-Pairs

- An alternate characterization of the two-pair is in terms of its chain parameters as

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Y_2 \\ X_2 \end{bmatrix}$$

where the matrix Γ given by

$$\Gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is called the **chain matrix** of the two-pair

Digital Two-Pairs

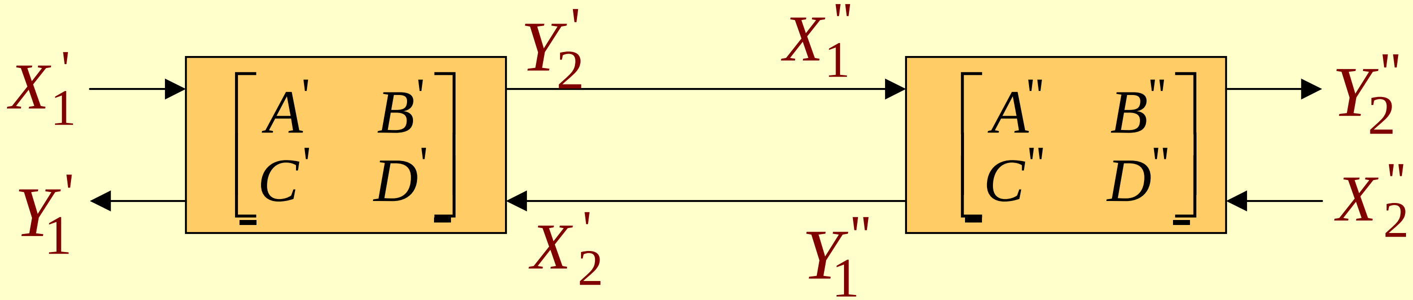
- The relation between the transfer parameters and the chain parameters are given by

$$t_{11} = \frac{C}{A}, \quad t_{12} = \frac{AD - BC}{A}, \quad t_{21} = \frac{1}{A}, \quad t_{22} = -\frac{C}{A}$$

$$A = \frac{1}{t_{21}}, \quad B = -\frac{t_{22}}{t_{21}}, \quad C = \frac{t_{11}}{t_{21}}, \quad D = \frac{t_{12}t_{21} - t_{11}t_{22}}{t_{21}}$$

Two-Pair Interconnection Schemes

Cascade Connection - Γ-cascade



- Here

$$\begin{bmatrix} X_1' \\ Y_1' \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} Y_2' \\ X_2' \end{bmatrix}$$

$$\begin{bmatrix} X_1'' \\ Y_1'' \end{bmatrix} = \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{bmatrix} Y_2'' \\ X_2'' \end{bmatrix}$$

Two-Pair Interconnection Schemes

- But from figure, $X_1^{''} = Y_2^{'}$ and $Y_1^{''} = X_2^{'}$
- Substituting the above relations in the first equation on the previous slide and combining the two equations we get

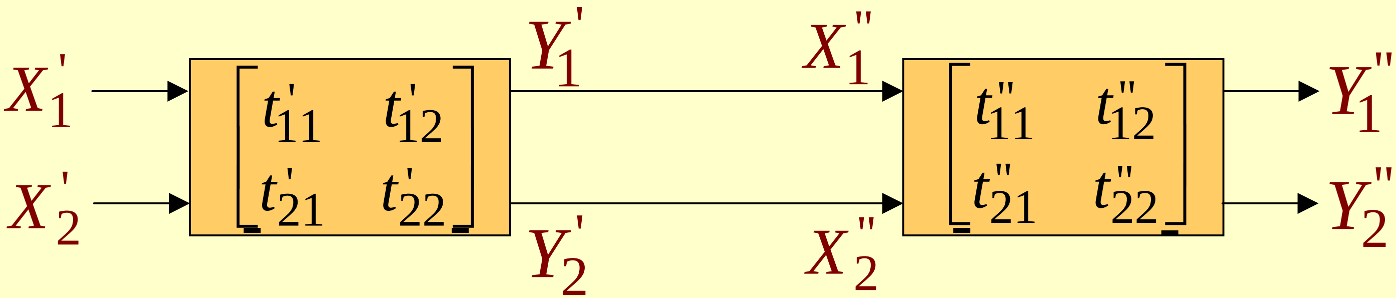
$$\begin{bmatrix} X_1^{'} \\ Y_1^{'} \end{bmatrix} = \begin{bmatrix} A^{'} & B^{'} \\ C^{'} & D^{'} \end{bmatrix} \begin{bmatrix} A^{''} & B^{''} \\ C^{''} & D^{''} \end{bmatrix} \begin{bmatrix} Y_2^{''} \\ X_2^{''} \end{bmatrix}$$

- Hence,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^{'} & B^{'} \\ C^{'} & D^{'} \end{bmatrix} \begin{bmatrix} A^{''} & B^{''} \\ C^{''} & D^{''} \end{bmatrix}$$

Two-Pair Interconnection Schemes

Cascade Connection - τ-cascade



- Here $\begin{bmatrix} Y_1' \\ Y_2' \end{bmatrix} = \begin{bmatrix} t_{11}' & t_{12}' \\ t_{21}' & t_{22}' \end{bmatrix} \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$

$$\begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} = \begin{bmatrix} t_{11}'' & t_{12}'' \\ t_{21}'' & t_{22}'' \end{bmatrix} \begin{bmatrix} X_1'' \\ X_2'' \end{bmatrix}$$

Two-Pair Interconnection Schemes

- But from figure, $X_1^{''} = Y_1^{'}$ and $X_2^{''} = Y_2^{'}$
- Substituting the above relations in the first equation on the previous slide and combining the two equations we get

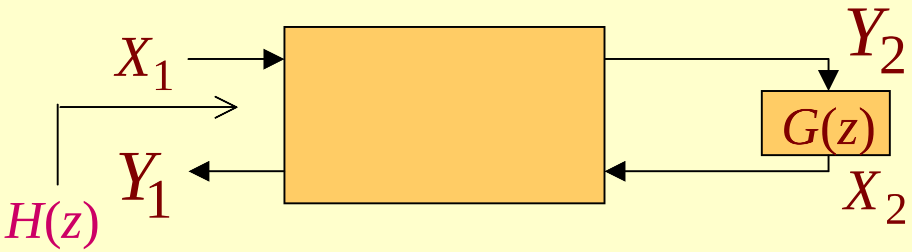
$$\begin{bmatrix} Y_1^{''} \\ Y_2^{''} \end{bmatrix} = \begin{bmatrix} t_{11}^{''} & t_{12}^{''} \\ t_{21}^{''} & t_{22}^{''} \end{bmatrix} \begin{bmatrix} t_{11}^{'} & t_{12}^{'} \\ t_{21}^{'} & t_{22}^{'} \end{bmatrix} \begin{bmatrix} X_1^{'} \\ X_2^{'} \end{bmatrix}$$

- Hence,

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11}^{''} & t_{12}^{''} \\ t_{21}^{''} & t_{22}^{''} \end{bmatrix} \begin{bmatrix} t_{11}^{'} & t_{12}^{'} \\ t_{21}^{'} & t_{22}^{'} \end{bmatrix}$$

Two-Pair Interconnection Schemes

Constrained Two-Pair



- It can be shown that

$$H(z) = \frac{Y_1}{X_1} = \frac{C + D \cdot G(z)}{A + B \cdot G(z)}$$

$$= t_{11} + \frac{t_{12} t_{21} G(z)}{1 - t_{22} G(z)}$$

Algebraic Stability Test

- We have shown that the BIBO stability of a causal rational transfer function requires that all its poles be inside the unit circle
- For very high-order transfer functions, it is very difficult to determine the pole locations analytically
- Root locations can of course be determined on a computer by some type of root finding algorithms

Algebraic Stability Test

- We now outline a simple algebraic test that does not require the determination of pole locations

The Stability Triangle

- For a 2nd-order transfer function the stability can be easily checked by examining its denominator coefficients

Algebraic Stability Test

- Let

$$D(z) = 1 + d_1 z^{-1} + d_2 z^{-2}$$

denote the denominator of the transfer function

- In terms of its poles, $D(z)$ can be expressed as

$$D(z) = (1 - \lambda_1 z^{-1})(1 - \lambda_2 z^{-1}) = 1 - (\lambda_1 + \lambda_2) z^{-1} + \lambda_1 \lambda_2 z^{-2}$$

- Comparing the last two equations we get

$$d_1 = -(\lambda_1 + \lambda_2), \qquad d_2 = \lambda_1 \lambda_2$$

Algebraic Stability Test

- The poles are inside the unit circle if

$$|\lambda_1| < 1, \quad |\lambda_2| < 1$$

- Now the coefficient d_2 is given by the product of the poles
- Hence we must have

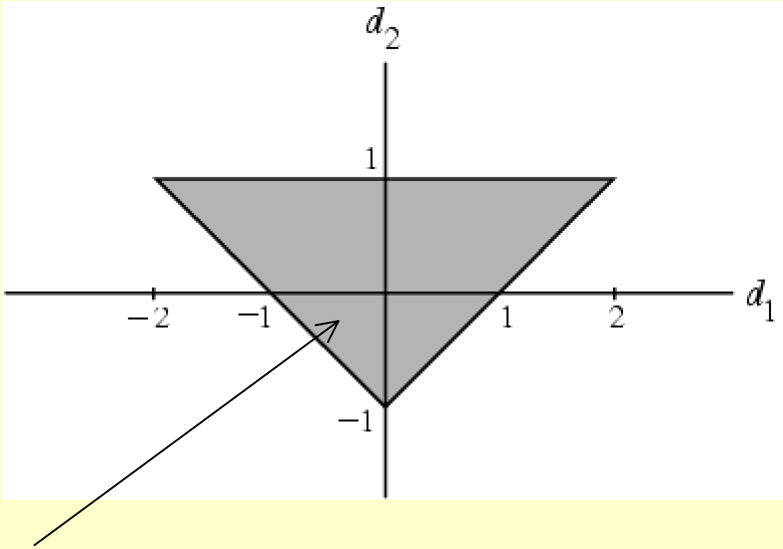
$$|d_2| < 1$$

- It can be shown that the second coefficient condition is given by

$$|d_1| < 1 + d_2$$

Algebraic Stability Test

- The region in the (d_1, d_2)-plane where the two coefficient condition are satisfied, called the stability triangle, is shown below



Algebraic Stability Test

- Example - Consider the two 2nd-order bandpass transfer functions designed earlier:

$$H'_{BP}(z) = -0.18819 \frac{1 - z^{-2}}{1 - 0.7343424 z^{-1} + 1.37638 z^{-2}}$$

$$H''_{BP}(z) = 0.13673 \frac{1 - z^{-2}}{1 - 0.533531 z^{-1} + 0.72654253 z^{-2}}$$

Algebraic Stability Test

- In the case of $H'_{BP}(z)$, we observe that

$$d_1 = -0.7343424, \qquad d_2 = 1.3763819$$

- Since here $|d_2| > 1$, $H'_{BP}(z)$ is unstable
- On the other hand, in the case of $H''_{BP}(z)$, we observe that

$$d_1 = -0.53353098, \qquad d_2 = 0.726542528$$

- Here, $|d_2| < 1$ and $|d_1| < 1 + d_2$, and hence $H''_{BP}(z)$ is BIBO stable

Algebraic Stability Test

A General Stability Test Procedure

- Let $D_M(z)$ denote the denominator of an M-th order causal IIR transfer function $H(z)$:

$$D_M(z) = \sum_{i=0}^{M} d_i z^{-i}$$

  where we assume $d_0 = 1$ for simplicity

- Define an M-th order allpass transfer function:

$$A_M(z) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

Algebraic Stability Test

- Or, equivalently

$$A_M(z) = \frac{d_M + d_{M-1}z^{-1} + d_{M-2}z^{-2} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

- If we express

$$D_M(z) = \prod_{i=1}^{M} (1 - \lambda_i z^{-i})$$

then it follows that

$$d_M = (-1)^M \prod_{i=1}^{M} \lambda_i$$

Algebraic Stability Test

- Now for stability we must have $|\lambda_i| < 1$, which implies the condition $|d_M| < 1$
- Define

$$k_M = A_M(\infty) = d_M$$

- Then a necessary condition for stability of $A_M(z)$, and hence, the transfer function $H(z)$ is given by

$$k_M^2 < 1$$

Algebraic Stability Test

- Assume the above condition holds
- We now form a new function

$$A_{M-1}(z) = z\left[\frac{A_M(z) - k_M}{1 - k_M A_M(z)}\right] = z\left[\frac{A_M(z) - d_M}{1 - d_M A_M(z)}\right]$$

- Substituting the rational form of $A_M(z)$ in the above equation we get

$$A_{M-1}(z) = \frac{d'_{M-1} + d'_{M-2} z^{-1} + \cdots + d'_1 z^{-(M-2)} + z^{-(M-1)}}{1 + d'_1 z^{-1} + \cdots + d'_{M-2} z^{-(M-2)} + d'_{M-1} z^{-(M-1)}}$$

Algebraic Stability Test

where

$$d_i' = \frac{d_i - d_M d_{M-i}}{1 - d_M^2}, \quad 1 \le i \le M-1$$

- Hence, $A_{M-1}(z)$ is an allpass function of order $M-1$
- Now the poles λ_o of $A_{M-1}(z)$ are given by the roots of the equation

$$A_M(\lambda_o) = \frac{1}{k_M}$$

Algebraic Stability Test

- By assumption $k_M^2 < 1$
- Hence $|A_M(\lambda_o)| > 1$
- If $A_M(z)$ is a stable allpass function, then

$$|A_M(z)| \begin{cases} <1, & \text{for } |z| > 1 \\ =1, & \text{for } |z| = 1 \\ >1, & \text{for } |z| < 1 \end{cases}$$

- Thus, if $A_M(z)$ is a stable allpass function, then the condition $|A_M(\lambda_o)| > 1$ holds only if

$$|\lambda_o| < 1$$

Algebraic Stability Test

- Or, in other words $A_{M-1}(z)$ is a stable allpass function
- Thus, if $A_M(z)$ is a stable allpass function and $k_M^2 < 1$, then $A_{M-1}(z)$ is also a stable allpass function of one order lower
- We now prove the converse, i.e., if $A_{M-1}(z)$ is a stable allpass function and $k_M^2 < 1$, then $A_M(z)$ is also a stable allpass function

Algebraic Stability Test

- To this end, we express $A_M(z)$ in terms of $A_{M-1}(z)$ arriving at

$$A_M(z) = \frac{k_M + z^{-1}A_{M-1}(z)}{1 + k_M z^{-1}A_{M-1}(z)}$$

- If ζ_o is a pole of $A_M(z)$, then

$$\zeta_o^{-1}A_{M-1}(\zeta_o) = -\frac{1}{k_M}$$

- By assumption $k_M^2 < 1$ holds

Algebraic Stability Test

- Therefore, $|\zeta_o^{-1} A_{M-1}(\zeta_o)| > 1$ i.e.,

  $$|A_{M-1}(\zeta_o)| > |\zeta_o|$$

- Assume $A_{M-1}(z)$ is a stable allpass function
- Then $|A_{M-1}(z)| \leq 1$ for $|z| \geq 1$
- Now, if $|\zeta_o| \geq 1$, then because of the above condition $|A_{M-1}(\zeta_o)| \leq 1$
- But the condition $|A_{M-1}(\zeta_o)| > |\zeta_o|$ reduces to $|A_{M-1}(\zeta_o)| > 1$ if $|\zeta_o| \geq 1$

Algebraic Stability Test

- Thus there is a contradiction
- On the other hand, if $|\zeta_o| < 1$ then from

$$|A_{M-1}(z)| > 1 \qquad \text{for } |z| < 1$$

  we have $|A_{M-1}(\zeta_o)| > 1$
- The above condition does not violate the condition $|A_{M-1}(\zeta_o)| > |\zeta_o|$

Algebraic Stability Test

- Thus, if $k_M^2 < 1$ and if $A_{M-1}(z)$ is a stable allpass function, then $A_M(z)$ is also a stable allpass function
- Summarizing, a necessary and sufficient set of conditions for the causal allpass function $A_M(z)$ to be stable is therefore:

  (1) $k_M^2 < 1$, and

  (2) The allpass function $A_{M-1}(z)$ is stable

Algebraic Stability Test

- Thus, once we have checked the condition $k_M^2 < 1$, we test next for the stability of the lower-order allpass function $A_{M-1}(z)$
- The process is then repeated, generating a set of coefficients:

$$k_M, k_{M-1}, \ldots, k_2, k_1$$

and a set of allpass functions of decreasing order:

$$A_M(z), A_{M-1}(z), \ldots, A_2(z), A_1(z), A_0(z) = 1$$

Algebraic Stability Test

- The allpass function $A_M(z)$ is stable if and only if $k_i^2 < 1$ for i
- Example - Test the stability of

$$H(z) = \frac{1}{4z^4 + 3z^3 + 2z^2 + z + 1}$$

- From $H(z)$ we generate a 4-th order allpass function

$$A_4(z) = \frac{\frac{1}{4}z^4 + \frac{1}{4}z^3 + \frac{1}{2}z^2 + \frac{3}{4}z + 1}{z^4 + \frac{3}{4}z^3 + \frac{1}{2}z^2 + \frac{1}{4}z + \frac{1}{4}} = \frac{d_4 z^4 + d_3 z^3 + d_2 z^2 + d_1 z + 1}{z^4 + d_1 z^3 + d_2 z^2 + d_3 z + d_4}$$

- Note: $k_4 = A_4(\infty) = d_4 = \frac{1}{4} < 1$

Algebraic Stability Test

- Using

$$d_i^{'} = \frac{d_i - d_4 d_{4-i}}{1 - d_4^2}, \quad 1 \le i \le 3$$

we determine the coefficients $\{d_i^{'}\}$ of the third-order allpass function $A_3(z)$ from the coefficients $\{d_i\}$ of $A_4(z)$:

$$A_3(z) = \frac{d_3^{'} z^3 + d_2^{'} z^2 + d_1^{'} z + 1}{d_1^{'} z^3 + d_2^{'} z^2 + d_3^{'} z + 1} = \frac{\frac{1}{15} z^3 + \frac{2}{5} z^2 + \frac{11}{15} z + 1}{z^3 + \frac{11}{15} z^2 + \frac{2}{5} z + \frac{1}{15}}$$

Algebraic Stability Test

- Note: $k_3 = A_3(\infty) = d_3' = \frac{1}{15} < 1$
- Following the above procedure, we derive the next two lower-order allpass functions:

$$A_2(z) = \frac{\frac{79}{224} z^2 + \frac{159}{224} z + 1}{z^2 + \frac{159}{224} z + \frac{79}{224}}$$

$$A_1(z) = \frac{\frac{53}{101} z + 1}{z + \frac{53}{101}}$$

Algebraic Stability Test

- Note: $k_2 = A_2(\infty) = \frac{79}{224} < 1$

  $k_1 = A_1(\infty) = \frac{53}{101} < 1$

- Since all of the stability conditions are satisfied, $A_4(z)$ and hence $H(z)$ are stable
- Note: It is not necessary to derive $A_3(z)$ since $A_2(z)$ can be tested for stability using the coefficient conditions