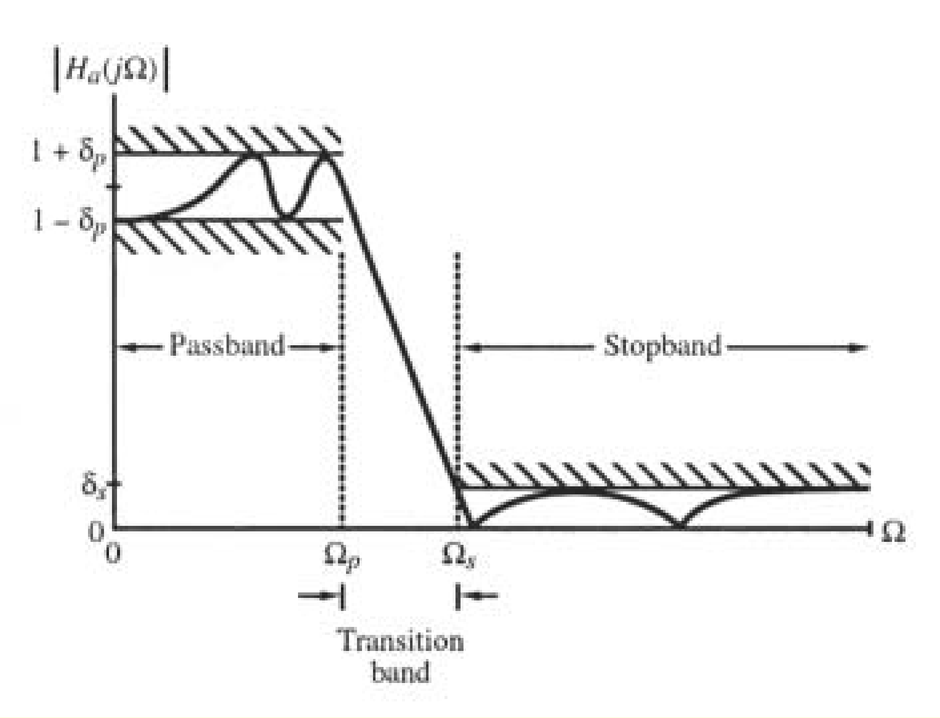
Analog Lowpass Filter Specifications

- Typical magnitude response $\left|H_a(j\Omega)\right|$ of an analog lowpass filter may be given as indicated below



Analog Lowpass Filter Specifications

- In the **passband**, defined by $0 \le \Omega \le \Omega_p$, we require
$$1-\delta_p \le |H_a(j\Omega)| \le 1+\delta_p, \quad |\Omega| \le \Omega_p$$
i.e., $|H_a(j\Omega)|$ approximates unity within an error of $\pm\delta_p$
- In the **stopband**, defined by $\Omega_s \le \Omega \le \infty$, we require
$$|H_a(j\Omega)| \le \delta_s, \quad \Omega_s \le |\Omega| \le \infty$$
i.e., $|H_a(j\Omega)|$ approximates zero within an error of δ_s

Analog Lowpass Filter Specifications

- Ω_p - **passband edge frequency**
- Ω_s - **stopband edge frequency**
- δ_p - **peak ripple value** in the passband
- δ_s - **peak ripple value** in the stopband
- **Peak passband ripple**

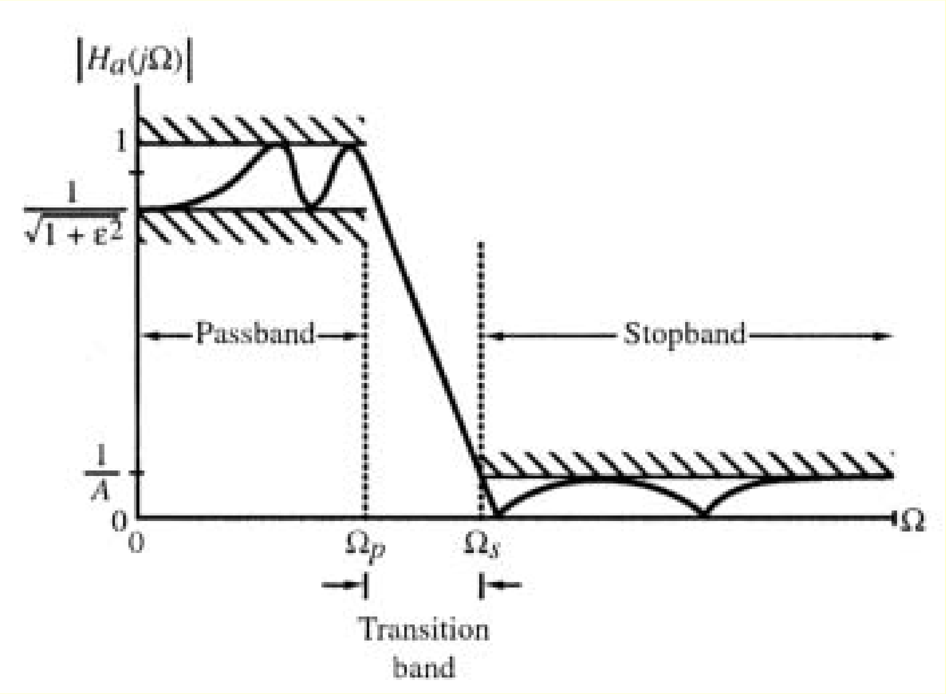
$$\alpha_p = -20\log_{10}(1-\delta_p) \text{ dB}$$

- **Minimum stopband attenuation**

$$\alpha_s = -20\log_{10}(\delta_s) \text{ dB}$$

Analog Lowpass Filter Specifications

- Magnitude specifications may alternately be given in a normalized form as indicated below



Analog Lowpass Filter Specifications

- Here, the maximum value of the magnitude in the passband assumed to be unity

- $1/\sqrt{1+\varepsilon^2}$ - Maximum passband deviation, given by the minimum value of the magnitude in the passband

- $\dfrac{1}{A}$ - Maximum stopband magnitude

Analog Lowpass Filter Design

- Two additional parameters are defined -

(1) **Transition ratio** $k = \dfrac{\Omega_p}{\Omega_s}$

For a lowpass filter $k < 1$

(2) **Discrimination parameter** $k_1 = \dfrac{\varepsilon}{\sqrt{A^2 - 1}}$

Usually $k_1 << 1$

Butterworth Approximation

- The magnitude-square response of an N-th order analog lowpass **Butterworth filter** is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

- First $2N-1$ derivatives of $|H_a(j\Omega)|^2$ at $\Omega = 0$ are equal to zero
- The Butterworth lowpass filter thus is said to have a **maximally-flat magnitude** at $\Omega = 0$

Butterworth Approximation

- Gain in dB is $\mathcal{G}(\Omega) = 10\log_{10}|H_a(j\Omega)|^2$

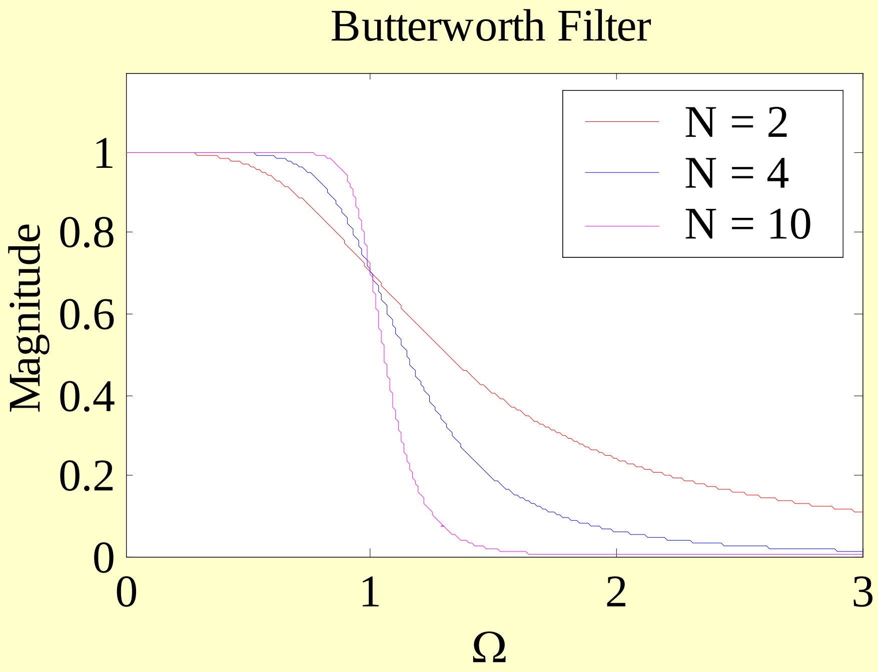
- As $\mathcal{G}(0) = 0$ and

  $\mathcal{G}(\Omega_c) = 10\log_{10}(0.5) = -3.0103 \cong -3$ dB

  Ω_c is called the **3-dB cutoff frequency**

Butterworth Approximation

- Typical magnitude responses with $\Omega_c = 1$



Butterworth Approximation

- Two parameters completely characterizing a Butterworth lowpass filter are Ω_c and N
- These are determined from the specified bandedges Ω_p and Ω_s, and minimum passband magnitude $1/\sqrt{1+\varepsilon^2}$, and maximum stopband ripple $1/A$

Butterworth Approximation

- Ω_c and N are thus determined from

$$\left|H_a(j\Omega_p)\right|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}$$

$$\left|H_a(j\Omega_s)\right|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = \frac{1}{A^2}$$

- Solving the above we get

$$N = \frac{1}{2}\cdot\frac{\log_{10}[(A^2-1)/\varepsilon^2]}{\log_{10}(\Omega_s/\Omega_p)} = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)}$$

Butterworth Approximation

- Since order N must be an integer, value obtained is rounded up to the next highest integer
- This value of N is used next to determine Ω_c by satisfying either the stopband edge or the passband edge specification exactly
- If the stopband edge specification is satisfied, then the passband edge specification is exceeded providing a safety margin

Butterworth Approximation

- Transfer function of an analog Butterworth lowpass filter is given by

$$H_a(s) = \frac{C}{D_N(s)} = \frac{\Omega_c^N}{s^N + \sum_{\ell=0}^{N-1} d_\ell s^\ell} = \frac{\Omega_c^N}{\prod_{\ell=1}^{N}(s - p_\ell)}$$

where

$$p_\ell = \Omega_c e^{j[\pi(N+2\ell-1)/2N]}, \;\; 1 \le \ell \le N$$

- Denominator $D_N(s)$ is known as the **Butterworth polynomial** of order N

Butterworth Approximation

- Example - Determine the lowest order of a Butterworth lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz

- Now
$$10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -1$$

which yields $\varepsilon^2 = 0.25895$

and
$$10\log_{10}\left(\frac{1}{A^2}\right) = -40$$

which yields $A^2 = 10{,}000$

Butterworth Approximation

- Therefore $$\frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\varepsilon} = 196.51334$$

  and $$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 5$$

- Hence
$$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.2811$$

- We choose $N = 4$

Chebyshev Approximation

- The magnitude-square response of an N-th order analog lowpass **Type 1 Chebyshev filter** is given by

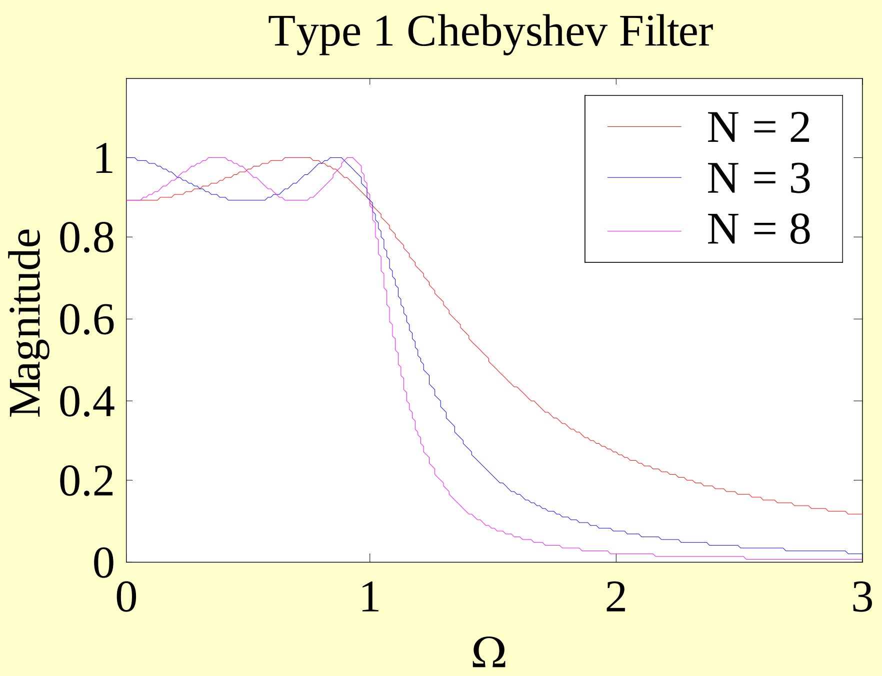
$$\left|H_a(s)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

where $T_N(\Omega)$ is the **Chebyshev polynomial** of order N:

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega), & |\Omega| > 1 \end{cases}$$

Chebyshev Approximation

- Typical magnitude response plots of the analog lowpass Type 1 Chebyshev filter are shown below



Chebyshev Approximation

- If at $\Omega = \Omega_s$ the magnitude is equal to $1/A$, then

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega_s/\Omega_p)} = \frac{1}{A^2}$$

- Solving the above we get

$$N = \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)}$$

- Order N is chosen as the nearest integer greater than or equal to the above value

Chebyshev Approximation

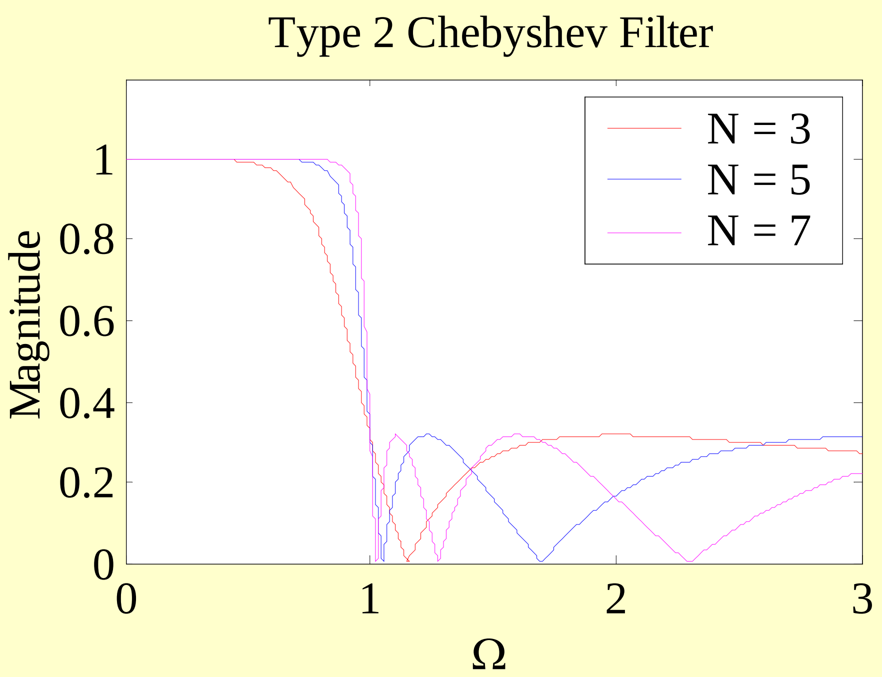
- The magnitude-square response of an N-th order analog lowpass **Type 2 Chebyshev** (also called **inverse Chebyshev) filter** is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}\right]^2}$$

where $T_N(\Omega)$ is the **Chebyshev polynomial** of order N

Chebyshev Approximation

- Typical magnitude response plots of the analog lowpass Type 2 Chebyshev filter are shown below



Chebyshev Approximation

- The order N of the Type 2 Chebyshev filter is determined from given ε, Ω_s, and A using

$$N = \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)}$$

- Example - Determine the lowest order of a Chebyshev lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz -

$$N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 2.6059$$

Elliptic Approximation

- The square-magnitude response of an elliptic lowpass filter is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 R_N^2(\Omega/\Omega_p)}$$

where $R_N(\Omega)$ is a rational function of order N satisfying $R_N(1/\Omega) = 1/R_N(\Omega)$, with the roots of its numerator lying in the interval $0 < \Omega < 1$ and the roots of its denominator lying in the interval $1 < \Omega < \infty$

Elliptic Approximation

- For given Ω_p, Ω_s, ε, and A, the filter order can be estimated using

$$N \cong \frac{2\log_{10}(4/k_1)}{\log_{10}(1/\rho)}$$

where $k' = \sqrt{1-k^2}$

$$\rho_0 = \frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}$$

$$\rho = \rho_0 + 2(\rho_0)^5 + 15(\rho_0)^9 + 150(\rho_0)^{13}$$

Elliptic Approximation

- Example - Determine the lowest order of a elliptic lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz

  Note: $k = 0.2$ and $1/k_1 = 196.5134$

- Substituting these values we get

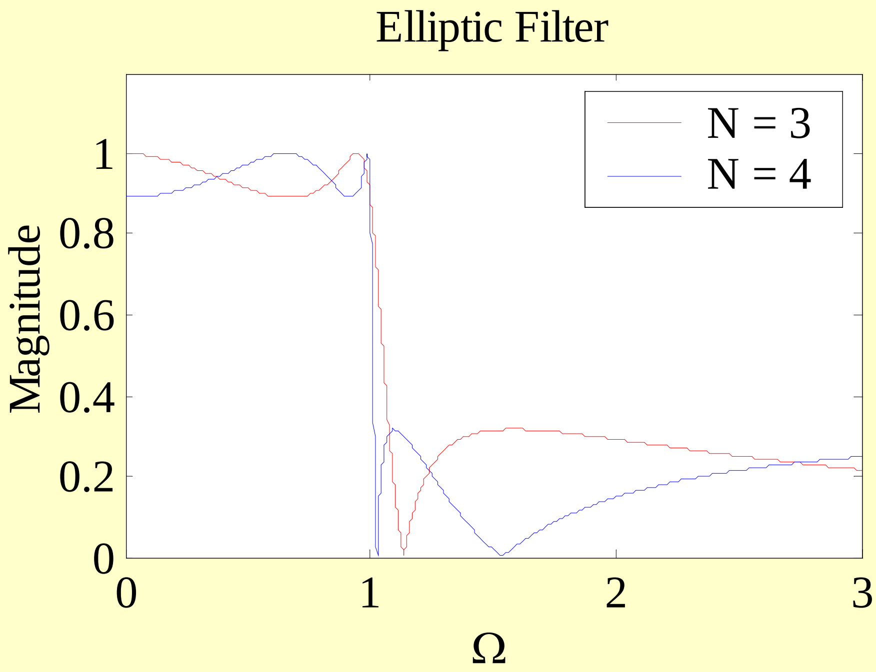
$$k' = 0.979796, \qquad \rho_0 = 0.00255135,$$

$$\rho = 0.0025513525$$

- and hence $N = 2.23308$
- Choose $N = 3$

Elliptic Approximation

- Typical magnitude response plots with $\Omega_p = 1$ are shown below



Analog Lowpass Filter Design

- Example - Design an elliptic lowpass filter of lowest order with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz
- Code fragments used

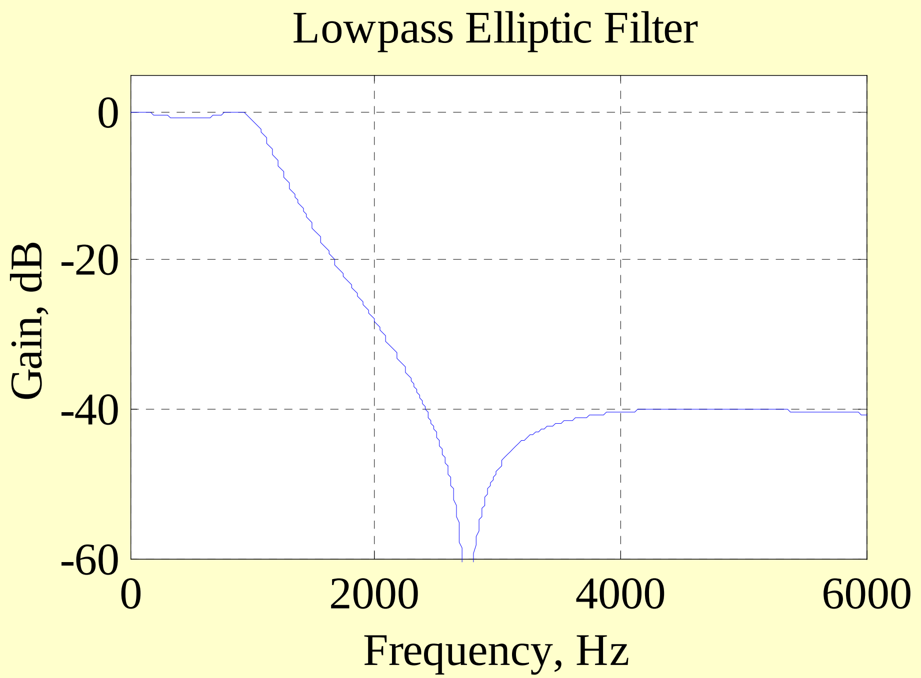
```
[N, Wn] = ellipord(Wp, Ws, Rp, Rs, 's');
[b, a] = ellip(N, Rp, Rs, Wn, 's');
```

with

```
Wp = 2*pi*1000;
Ws = 2*pi*5000;
Rp = 1;
Rs = 40;
```

Analog Lowpass Filter Design

- Gain plot



Design of Analog Highpass, Bandpass and Bandstop Filters

- Steps involved in the design process:

  Step 1 - Develop of specifications of a prototype analog lowpass filter $H_{LP}(s)$ from specifications of desired analog filter $H_D(s)$ using a frequency transformation

  Step 2 - Design the prototype analog lowpass filter

  Step 3 - Determine the transfer function $H_D(s)$ of desired analog filter by applying the inverse frequency transformation to $H_{LP}(s)$

Design of Analog Highpass, Bandpass and Bandstop Filters

- Let s denote the Laplace transform variable of prototype analog lowpass filter $H_{LP}(s)$ and $\hat{s}$ denote the Laplace transform variable of desired analog filter $H_D(\hat{s})$
- The mapping from s-domain to $\hat{s}$-domain is given by the invertible transformation

$$s = F(\hat{s})$$

- Then $H_D(\hat{s}) = H_{LP}(s)\big|_{s=F(\hat{s})}$

$$H_{LP}(s) = H_D(\hat{s})\big|_{\hat{s}=F^{-1}(s)}$$

Analog Highpass Filter Design

- Spectral Transformation:

$$s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$$

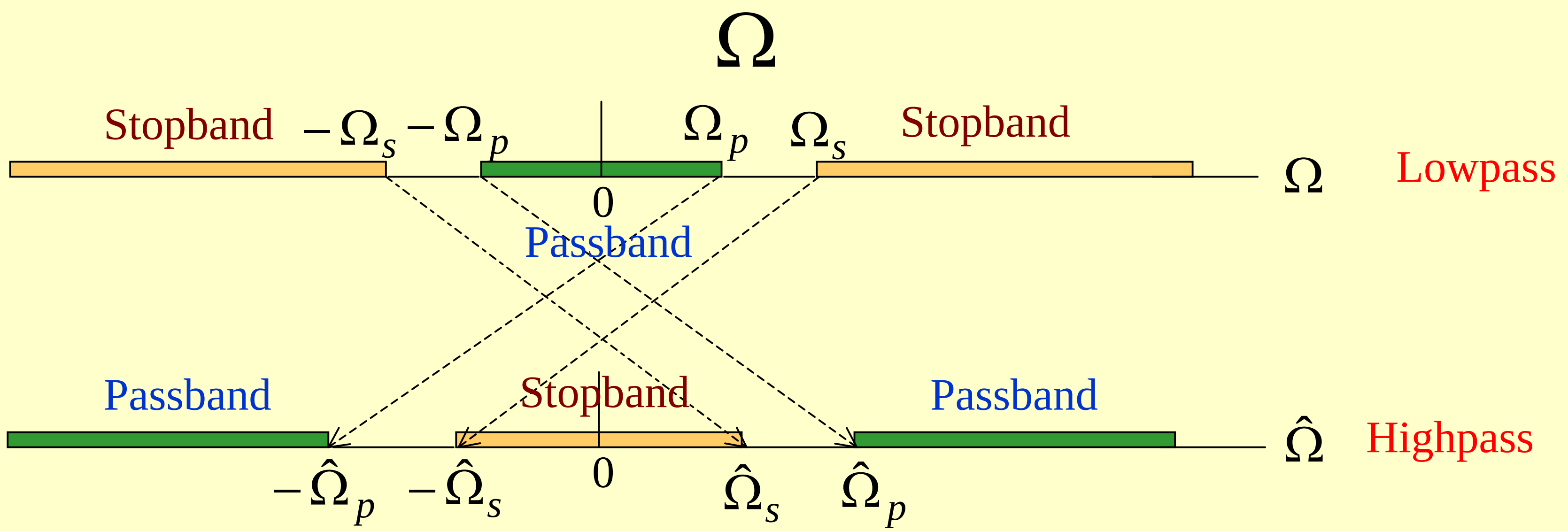
where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\hat{\Omega}_p$ is the passband edge frequency of $H_{HP}(\hat{s})$

- On the imaginary axis the transformation is

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

Analog Highpass Filter Design

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$



Analog Highpass Filter Design

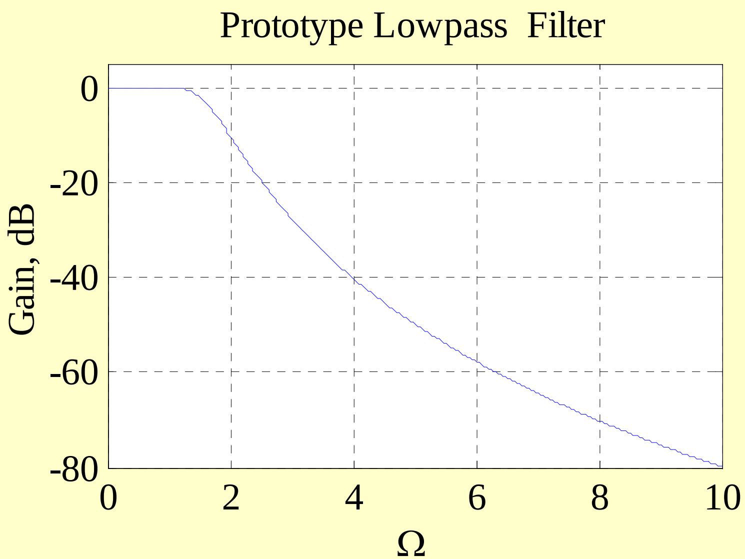
- Example - Design an analog Butterworth highpass filter with the specifications: $F_p = 4$ kHz, $F_s = 1$ kHz, $\alpha_p = 0.1$ dB, $\alpha_s = 40$ dB
- Choose $\Omega_p = 1$
- Then
$$\Omega_s = \frac{2\pi F_p}{2\pi F_s} = \frac{F_p}{F_s} = \frac{4000}{1000} = 4$$
- Analog lowpass filter specifications: $\Omega_p = 1$, $\Omega_s = 4$, $\alpha_p = 0.1$ dB, $\alpha_s = 40$ dB

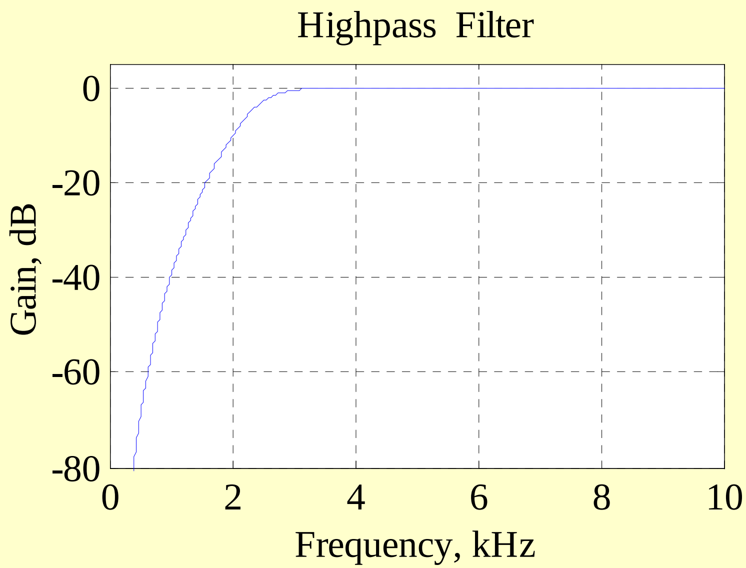
Analog Highpass Filter Design

- Code fragments used

```
[N, Wn] = buttord(1, 4, 0.1, 40, 's');
[B, A] = butter(N, Wn, 's');
[num, den] = lp2hp(B, A, 2*pi*4000);
```

- Gain plots





Analog Bandpass Filter Design

- Spectral Transformation

$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_o^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})}$$

where Ω_p is the passband edge frequency of $H_{LP}(s)$, and $\hat{\Omega}_{p1}$ and $\hat{\Omega}_{p2}$ are the lower and upper passband edge frequencies of desired bandpass filter $H_{BP}(\hat{s})$

Analog Bandpass Filter Design

- On the imaginary axis the transformation is

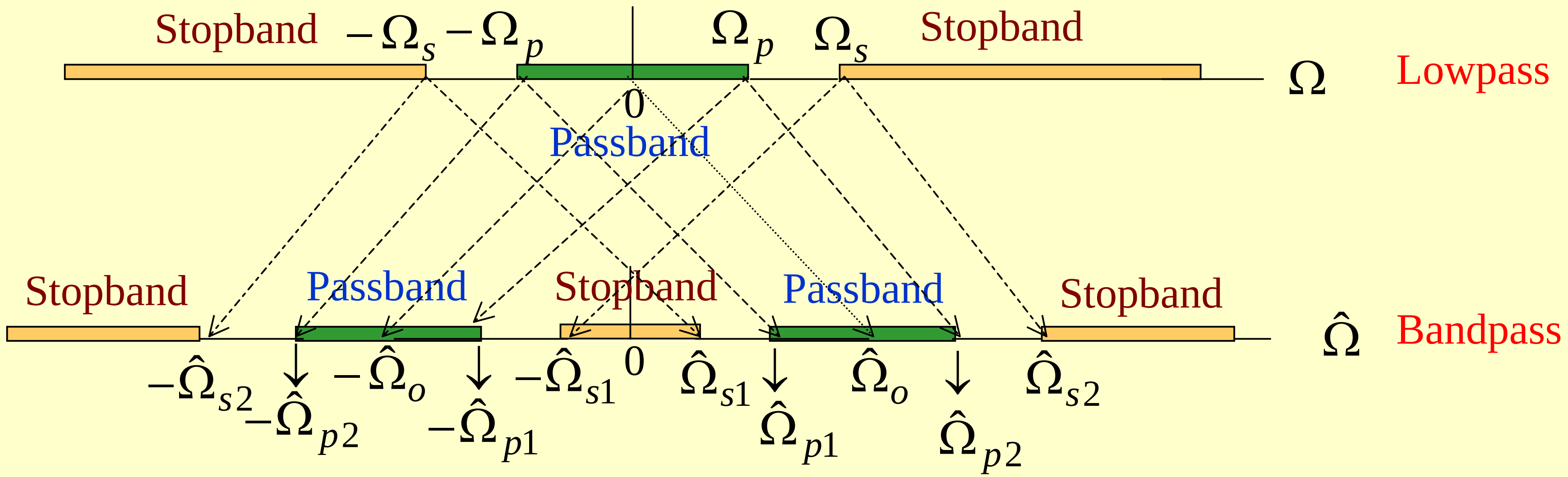
$$\Omega = -\Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega} B_w}$$

where $B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1}$ is the width of passband and $\hat{\Omega}_o$ is the **passband center frequency** of the bandpass filter

- Passband edge frequency $\pm\Omega_p$ is mapped into $\mp\hat{\Omega}_{p1}$ and $\pm\hat{\Omega}_{p2}$, lower and upper passband edge frequencies

Analog Bandpass Filter Design

$$\Omega = -\Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega} B_w}$$



Analog Bandpass Filter Design

- Stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\hat{\Omega}_{s1}$ and $\pm\hat{\Omega}_{s2}$, lower and upper stopband edge frequencies
- Also,

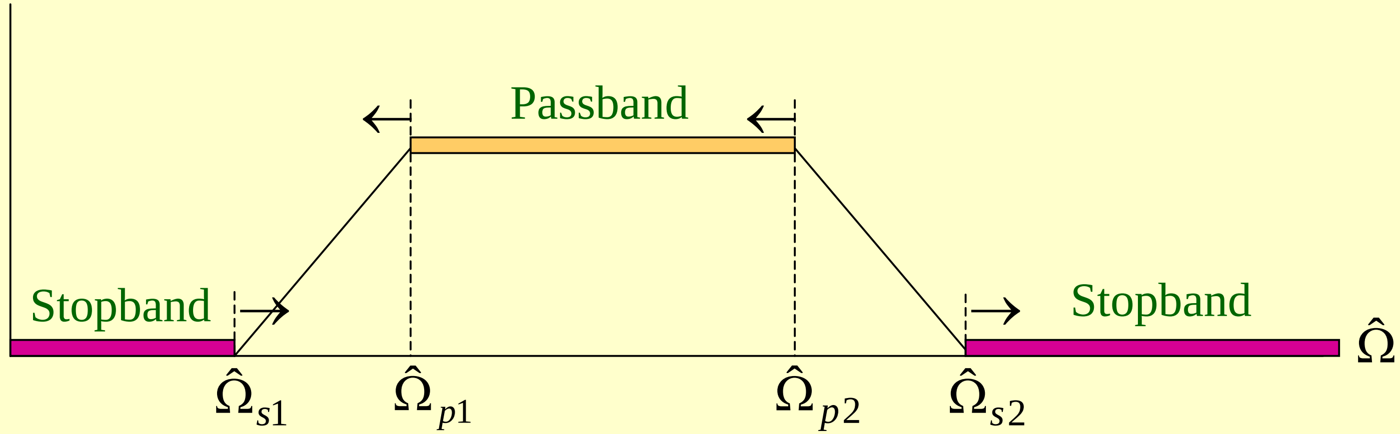
$$\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$$

- If bandedge frequencies do not satisfy the above condition, then one of the frequencies needs to be changed to a new value so that the condition is satisfied

Analog Bandpass Filter Design

- **Case 1**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} > \hat{\Omega}_{s1}\hat{\Omega}_{s2}$

  To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either increase any one of the stopband edges or decrease any one of the passband edges as shown below



Analog Bandpass Filter Design

(1) Decrease $\hat{\Omega}_{p1}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p2}$

⇒ larger passband and shorter leftmost transition band

(2) Increase $\hat{\Omega}_{s1}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s2}$

⇒ No change in passband and shorter leftmost transition band

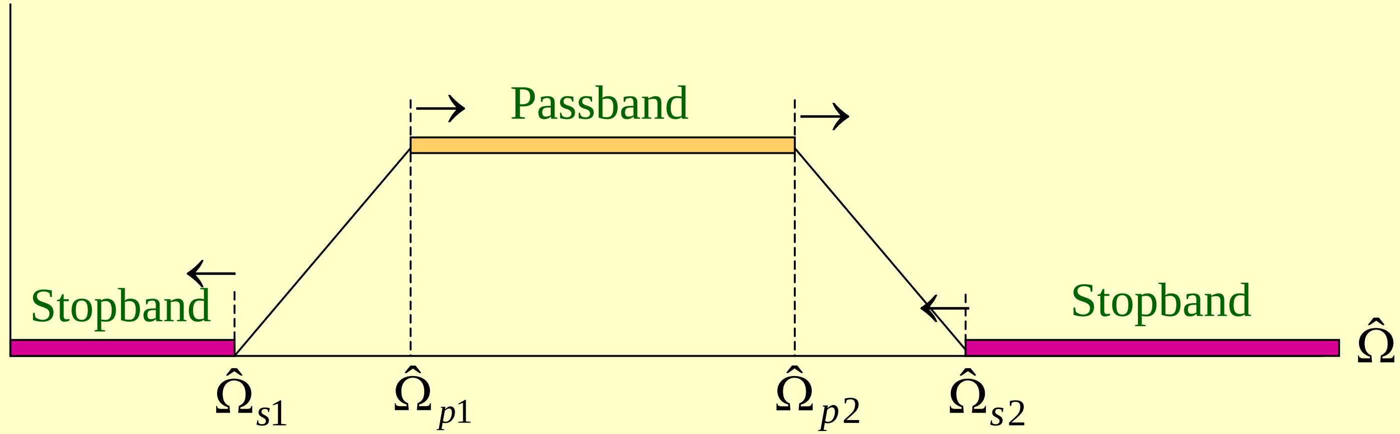
Analog Bandpass Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by decreasing $\hat{\Omega}_{p2}$ which is not acceptable as the passband is reduced from the desired value
- Alternately, the condition can be satisfied by increasing $\hat{\Omega}_{s2}$ which is not acceptable as the rightmost transition band is increased

Analog Bandpass Filter Design

- **Case 2**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} < \hat{\Omega}_{s1}\hat{\Omega}_{s2}$

  To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either decrease any one of the stopband edges or increase any one of the passband edges as shown below



Analog Bandpass Filter Design

(1) Increase $\hat{\Omega}_{p2}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p1}$

$\Rightarrow$ larger passband and shorter rightmost transition band

(2) Decrease $\hat{\Omega}_{s2}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s1}$

$\Rightarrow$ No change in passband and shorter rightmost transition band

Analog Bandpass Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by increasing $\hat{\Omega}_{p1}$ which is not acceptable as the passband is reduced from the desired value
- Alternately, the condition can be satisfied by decreasing $\hat{\Omega}_{s1}$ which is not acceptable as the leftmost transition band is increased

Analog Bandpass Filter Design

- Example - Design an analog elliptic bandpass filter with the specifications:
  $\hat{F}_{p1} = 4$ kHz, $\hat{F}_{p2} = 7$ kHz, $\hat{F}_{s1} = 3$ kHz
  $\hat{F}_{s2} = 8$ kHz, $\alpha_p = 1$ dB, $\alpha_s = 22$ dB

- Now $\hat{F}_{p1}\hat{F}_{p2} = 28 \times 10^6$ and $\hat{F}_{s1}\hat{F}_{s2} = 24 \times 10^6$
- Since $\hat{F}_{p1}\hat{F}_{p2} > \hat{F}_{s1}\hat{F}_{s2}$ we choose

$$\hat{F}_{p1} = \hat{F}_{s1}\hat{F}_{s2} / \hat{F}_{p2} = 3.571428 \text{ kHz}$$

Analog Bandpass Filter Design

- We choose $\Omega_p = 1$
- Hence

$$\Omega_s = \frac{24-9}{(25/7)\times 3} = 1.4$$

- Analog lowpass filter specifications: $\Omega_p = 1$,

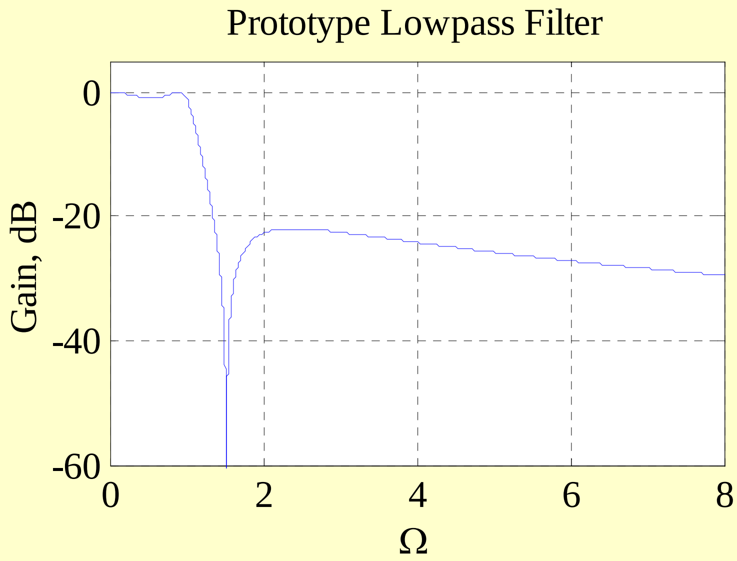
$\Omega_s = 1.4$, $\alpha_p = 1$ dB, $\alpha_s = 22$ dB

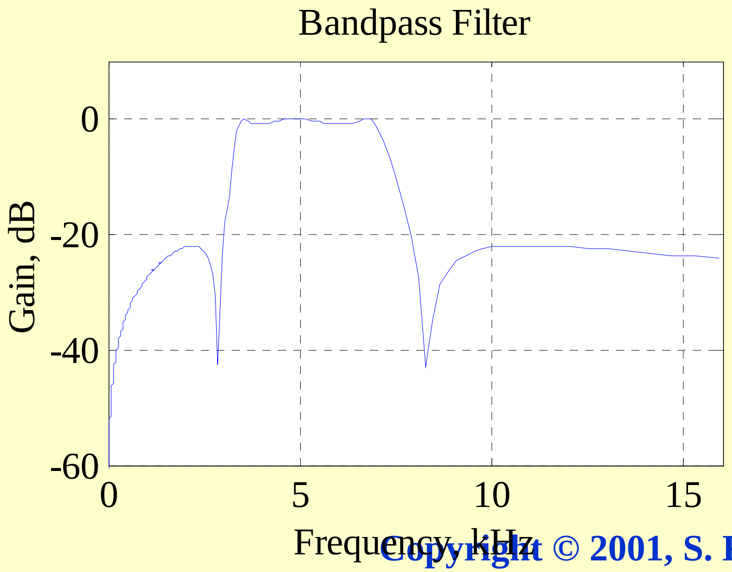
Analog Bandpass Filter Design

- Code fragments used

```
[N, Wn] = ellipord(1, 1.4, 1, 22, 's');
[B, A] = ellip(N, 1, 22, Wn, 's');
[num, den]
     = lp2bp(B, A, 2*pi*4.8989795, 2*pi*25/7);
```

- Gain plot





Analog Bandstop Filter Design

- Spectral Transformation

$$s = \Omega_s \frac{\hat{s}(\hat{\Omega}_{s2} - \hat{\Omega}_{s1})}{\hat{s}^2 + \hat{\Omega}^2}$$

where Ω_s is the stopband edge frequency of $H_{LP}(s)$, and $\hat{\Omega}_{s1}$ and $\hat{\Omega}_{s2}$ are the lower and upper stopband edge frequencies of the desired bandstop filter $H_{BS}(\hat{s})$

Analog Bandstop Filter Design

- On the imaginary axis the transformation is

$$\Omega = \Omega_s \frac{\hat{\Omega} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}^2}$$

where $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1}$ is the width of stopband and $\hat{\Omega}_o$ is the **stopband center frequency** of the bandstop filter

- Stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\hat{\Omega}_{s1}$ and $\pm\hat{\Omega}_{s2}$, lower and upper stopband edge frequencies

Analog Bandstop Filter Design

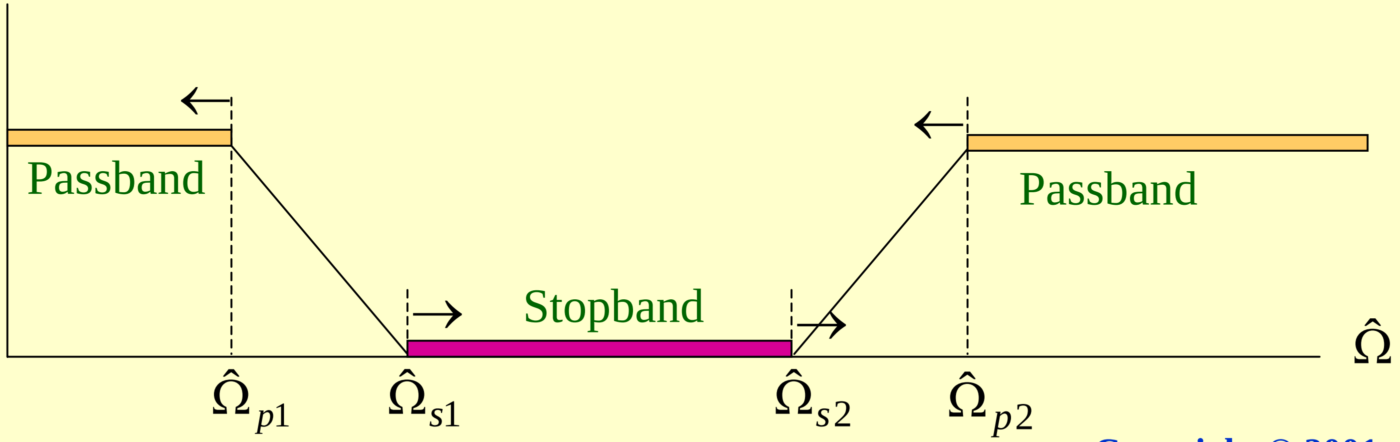
- Passband edge frequency $\pm\Omega_p$ is mapped into $\mp\hat{\Omega}_{p1}$ and $\pm\hat{\Omega}_{p2}$, lower and upper passband edge frequencies
- Also,

$$\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$$

- If bandedge frequencies do not satisfy the above condition, then one of the frequencies needs to be changed to a new value so that the condition is satisfied

Analog Bandstop Filter Design

- **Case 1**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} > \hat{\Omega}_{s1}\hat{\Omega}_{s2}$
- To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either increase any one of the stopband edges or decrease any one of the passband edges as shown below



Analog Bandstop Filter Design

(1) Decrease $\hat{\Omega}_{p2}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p2}$

$\Rightarrow$ larger high-frequency passband and shorter rightmost transition band

(2) Increase $\hat{\Omega}_{s2}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s2}$

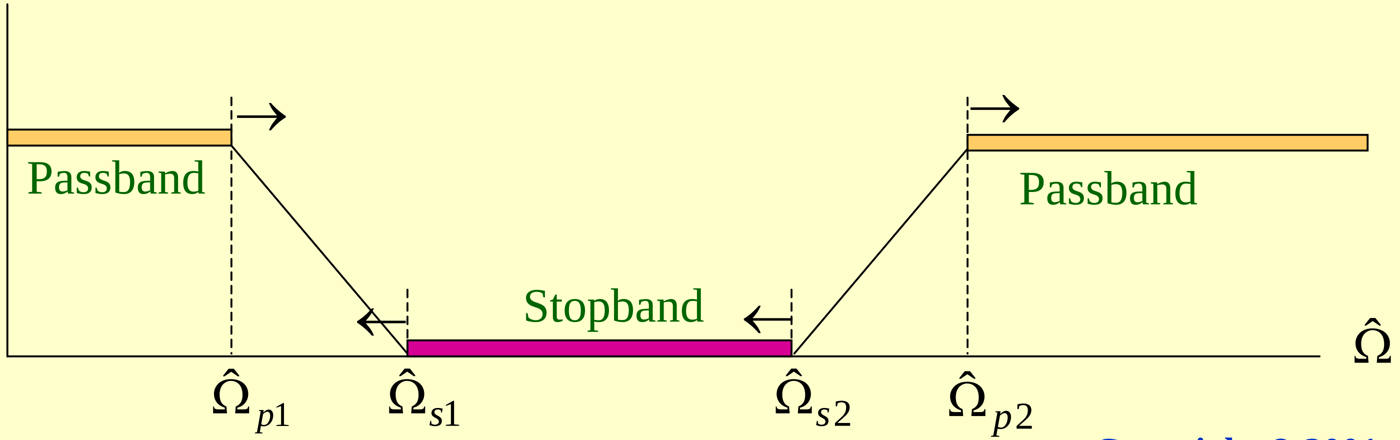
$\Rightarrow$ No change in passbands and shorter rightmost transition band

Analog Bandstop Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by decreasing $\hat{\Omega}_{p1}$ which is not acceptable as the low-frequency passband is reduced from the desired value
- Alternately, the condition can be satisfied by increasing $\hat{\Omega}_{s1}$ which is not acceptable as the leftmost transition band is increased

Analog Bandstop Filter Design

- **Case 1**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} < \hat{\Omega}_{s1}\hat{\Omega}_{s2}$
- To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either decrease any one of the stopband edges or increase any one of the passband edges as shown below



Analog Bandstop Filter Design

(1) Increase $\hat{\Omega}_{p1}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p1}$

$\Rightarrow$ larger passband and shorter leftmost transition band

(2) Decrease $\hat{\Omega}_{s1}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s1}$

$\Rightarrow$ No change in passbands and shorter leftmost transition band

Analog Bandstop Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by increasing $\hat{\Omega}_{p2}$ which is not acceptable as the high-frequency passband is decreased from the desired value
- Alternately, the condition can be satisfied by decreasing $\hat{\Omega}_{s2}$ which is not acceptable as the stopband is decreased