Fixed Window Functions

- Using a tapered window causes the height of the sidelobes to diminish, with a corresponding increase in the main lobe width resulting in a wider transition at the discontinuity
- Hann:

$$w[n] = 0.5 + 0.5\cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \le n \le M$$

- Hamming:

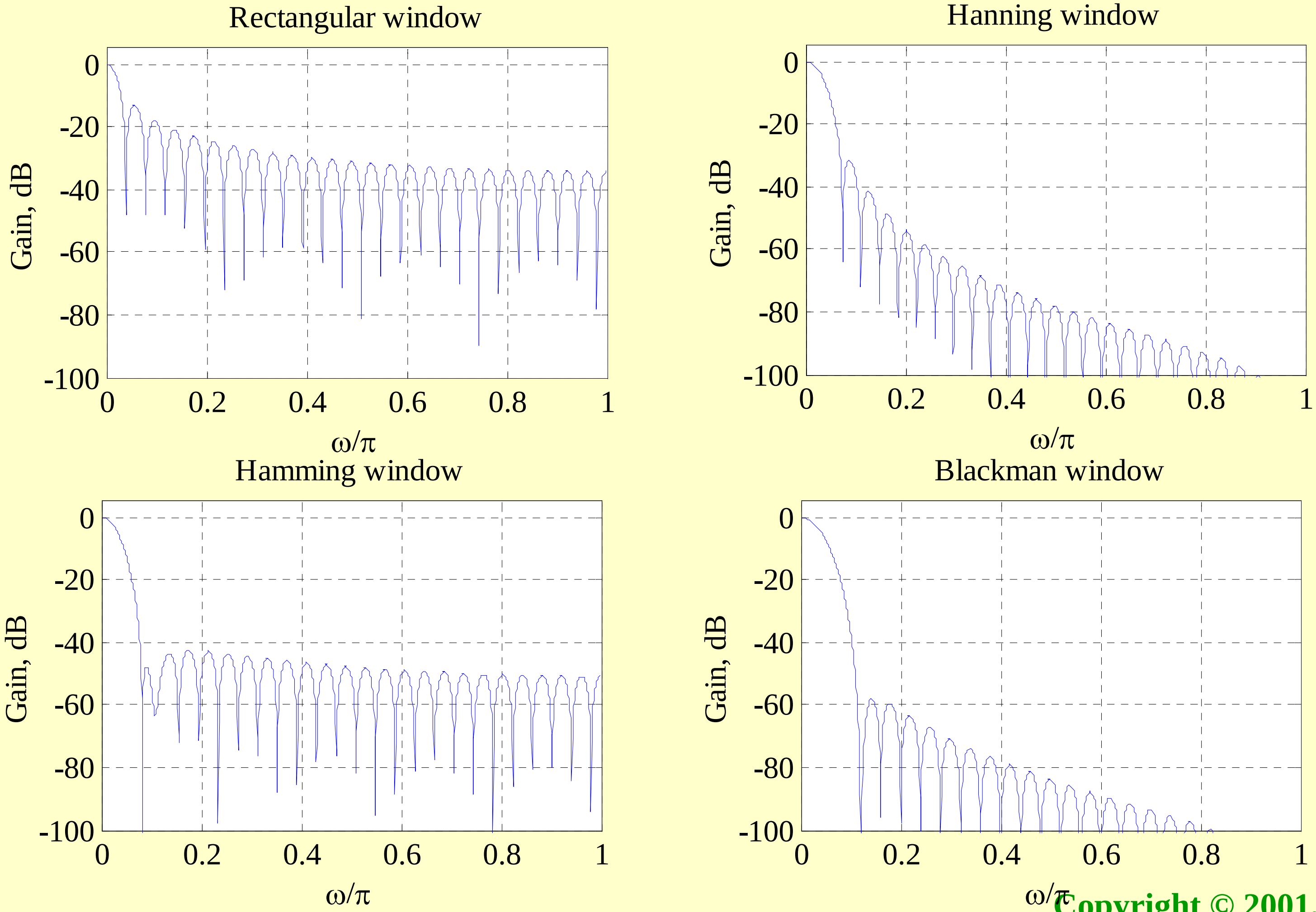
$$w[n] = 0.54 + 0.46\cos\left(\frac{2\pi n}{2M+1}\right), \quad -M \le n \le M$$

- Blackman:

$$w[n] = 0.42 + 0.5\cos\left(\frac{2\pi n}{2M+1}\right) + 0.08\cos\left(\frac{4\pi n}{2M+1}\right)$$

Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for $M = 25$ are shown below:



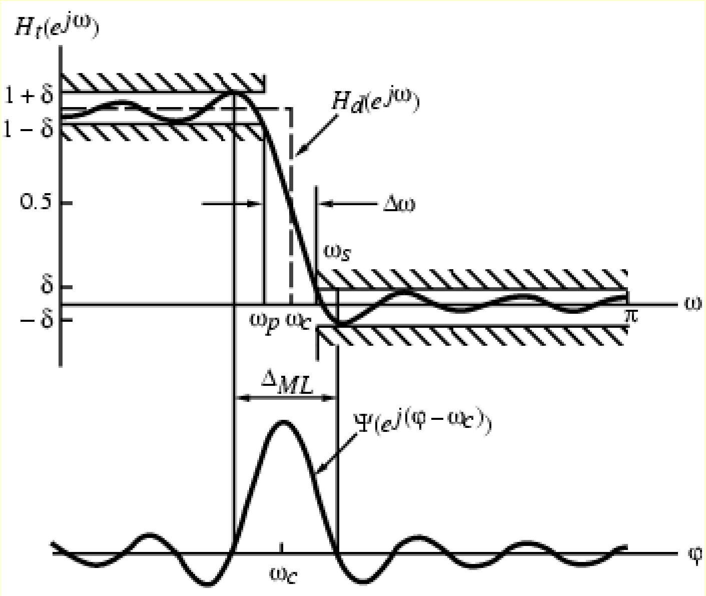
Fixed Window Functions

- Magnitude spectrum of each window characterized by a main lobe centered at $\omega = 0$ followed by a series of sidelobes with decreasing amplitudes
- Parameters predicting the performance of a window in filter design are:
- **Main lobe width**
- **Relative sidelobe level**

Fixed Window Functions

- **Main lobe width** Δ_{ML} - given by the distance between zero crossings on both sides of main lobe
- **Relative sidelobe level** $A_{s\ell}$ - given by the difference in dB between amplitudes of largest sidelobe and main lobe

Fixed Window Functions



- Observe $H_t(e^{j(\omega_c+\Delta\omega)}) + H_t(e^{j(\omega_c-\Delta\omega)}) \cong 1$
- Thus, $H_t(e^{j\omega_c}) \cong 0.5$
- Passband and stopband ripples are the same

Fixed Window Functions

- Distance between the locations of the maximum passband deviation and minimum stopband value $\cong \Delta_{ML}$

- Width of transition band

$$\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$$

Fixed Window Functions

- To ensure a fast transition from passband to stopband, window should have a very small main lobe width
- To reduce the passband and stopband ripple δ, the area under the sidelobes should be very small
- Unfortunately, these two requirements are contradictory

Fixed Window Functions

- In the case of rectangular, Hann, Hamming, and Blackman windows, the value of ripple does not depend on filter length or cutoff frequency ω_c, and is essentially constant
- In addition,

$$\Delta\omega \approx \frac{c}{M}$$

where c is a constant for most practical purposes

Fixed Window Functions

- **Rectangular window** - $\Delta_{ML} = 4\pi/(2M+1)$

  $A_{s\ell} = 13.3$ dB, $\alpha_s = 20.9$ dB, $\Delta\omega = 0.92\pi/M$

- **Hann window** - $\Delta_{ML} = 8\pi/(2M+1)$

  $A_{s\ell} = 31.5$ dB, $\alpha_s = 43.9$ dB, $\Delta\omega = 3.11\pi/M$

- **Hamming window** - $\Delta_{ML} = 8\pi/(2M+1)$

  $A_{s\ell} = 42.7$ dB, $\alpha_s = 54.5$ dB, $\Delta\omega = 3.32\pi/M$

- **Blackman window** - $\Delta_{ML} = 12\pi/(2M+1)$

  $A_{s\ell} = 58.1$ dB, $\alpha_s = 75.3$ dB, $\Delta\omega = 5.56\pi/M$

Fixed Window Functions

- Filter Design Steps -

(1) Set

$$\omega_c = (\omega_p + \omega_s)/2$$

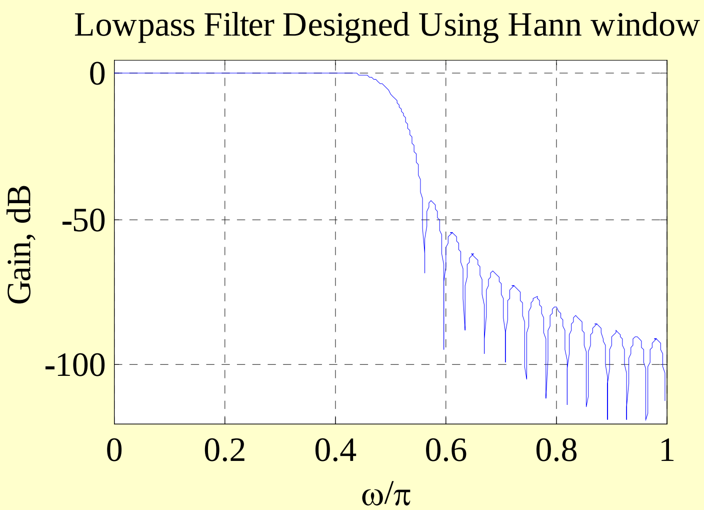
(2) Choose window based on specified α_s

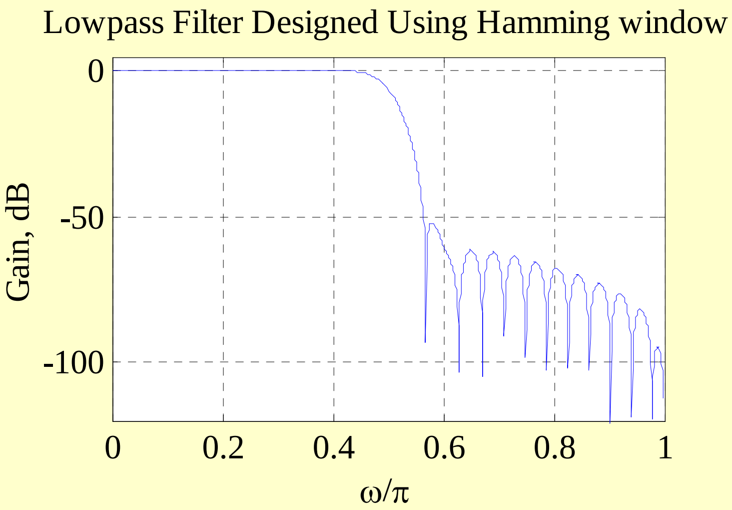
(3) Estimate M using

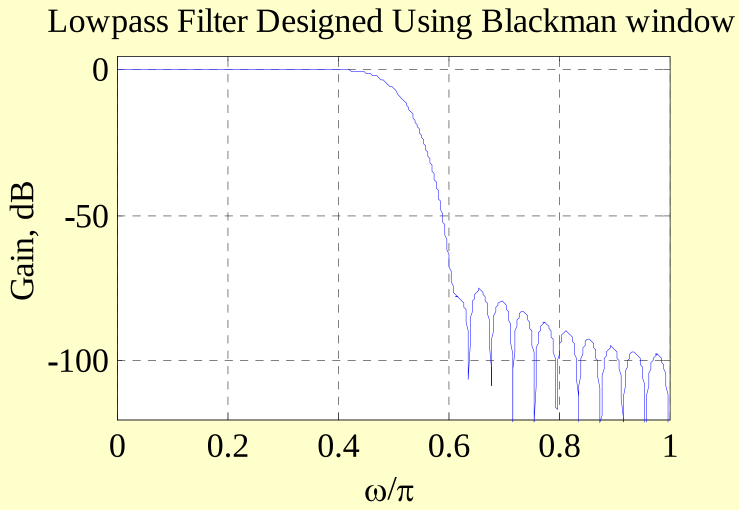
$$\Delta\omega \approx \frac{c}{M}$$

FIR Filter Design Example

- Lowpass filter of length 51 and $\omega_c = \pi/2$







FIR Filter Design Example

- An increase in the main lobe width is associated with an increase in the width of the transition band
- A decrease in the sidelobe amplitude results in an increase in the stopband attenuation

Adjustable Window Functions

- **Dolph-Chebyshev Window** -

$$w[n] = \frac{1}{2M+1}\left[\frac{1}{\gamma} + 2\sum_{k=1}^{M} T_k\left(\beta\cos\frac{k}{2M+1}\right)\cos\frac{2nk\pi}{2M+1}\right],$$
$$-M \le n \le M$$

where
$$\gamma = \frac{\text{amplitude of sidelobe}}{\text{main lobe amplitude}}$$
$$\beta = \cosh\left(\frac{1}{2M}\cosh^{-1}\frac{1}{\gamma}\right)$$

and
$$T_\ell(x) = \begin{cases} \cos(\ell\cos^{-1}x), & |x| \le 1 \\ \cosh(\ell\cosh^{-1}x), & |x| > 1 \end{cases}$$

Adjustable Window Functions

- Dolph-Chebyshev window can be designed with any specified relative sidelobe level while the main lobe width adjusted by choosing length appropriately
- Filter order is estimated using

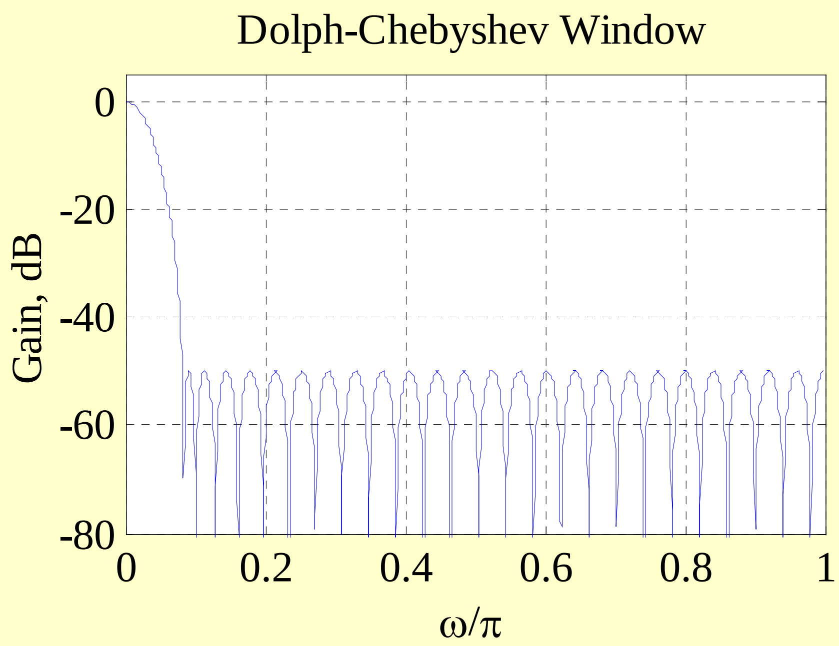
$$N = \frac{2.056\alpha_s - 16.4}{2.85(\Delta\omega)}$$

where $\Delta\omega$ is the normalized transition bandwidth, e.g, for a lowpass filter

$$\Delta\omega = \omega_s - \omega_p$$

Adjustable Window Functions

- Gain response of a Dolph-Chebyshev window of length 51 and relative sidelobe level of 50 dB is shown below



Adjustable Window Functions

Properties of Dolph-Chebyshev window:

- All sidelobes are of equal height
- Stopband approximation error of filters designed have essentially equiripple behavior
- For a given window length, it has the smallest main lobe width compared to other windows resulting in filters with the smallest transition band

Adjustable Window Functions

- **Kaiser Window** -

$$w[n] = \frac{I_0\{\beta\sqrt{1-(n/M)^2}\}}{I_0(\beta)}, \quad -M \le n \le M$$

where β is an adjustable parameter and $I_0(u)$ is the modified zeroth-order Bessel function of the first kind:

$$I_0(u) = 1 + \sum_{r=1}^{\infty}\left[\frac{(u/2)^r}{r!}\right]^2$$

- Note $I_0(u) > 0$ for $u > 0$
- In practice $I_0(u) \cong 1 + \sum_{r=1}^{20}\left[\frac{(u/2)^r}{r!}\right]^2$

Adjustable Window Functions

- β controls the minimum stopband attenuation of the windowed filter response
- β is estimated using

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \le \alpha_s \le 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

- Filter order is estimated using

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)}$$

where $\Delta\omega$ is the normalized transition bandwidth

FIR Filter Design Example

- Specifications: $\omega_p = 0.3\pi$, $\omega_s = 0.5\pi$, $\alpha_s = 40$ dB
- Thus $\omega_c = (\omega_p + \omega_s)/2 = 0.4\pi$

$$\delta_s = 10^{-\alpha_s/20} = 0.01$$

$$\beta = 0.5842(19)^{0.4} + 0.07886 \times 19 = 3.3953$$

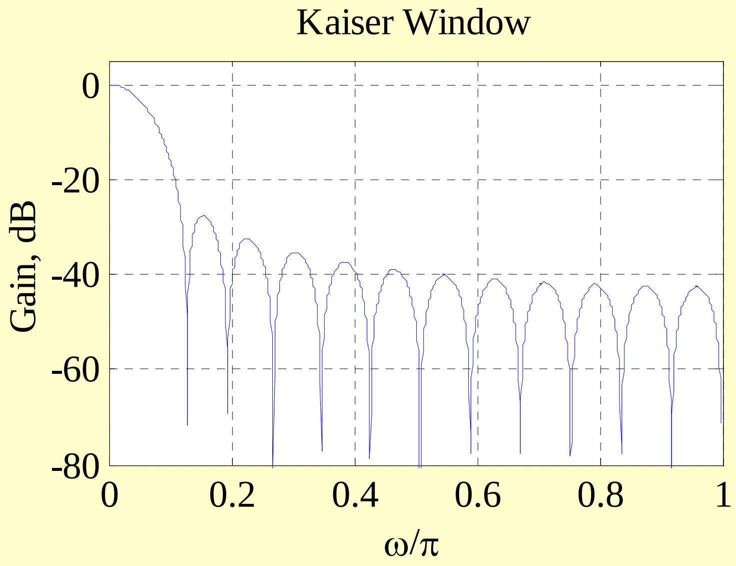
$$N = \frac{32}{2.285(0.2\pi)} = 22.2886$$

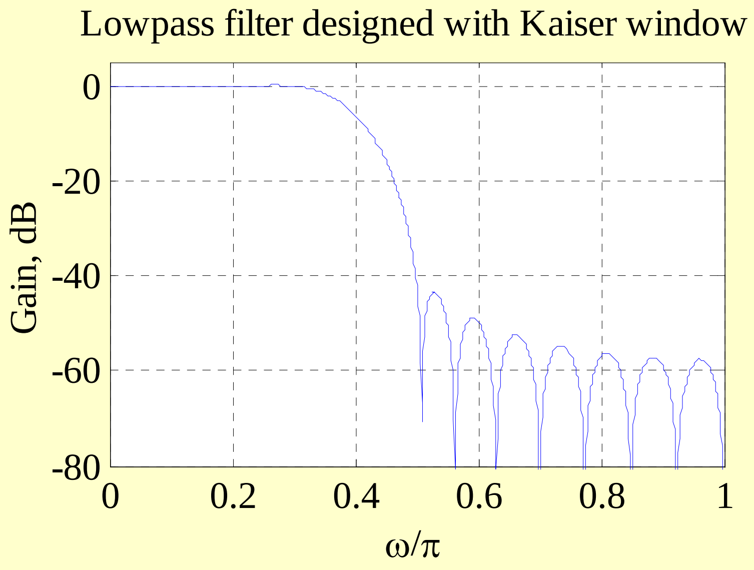
- Choose $N = 24$ implying $M = 12$

FIR Filter Design Example

- Hence $h_t[n] = \frac{\sin(0.4\pi n)}{\pi n} \cdot w[n], \quad -12 \le n \le 12$

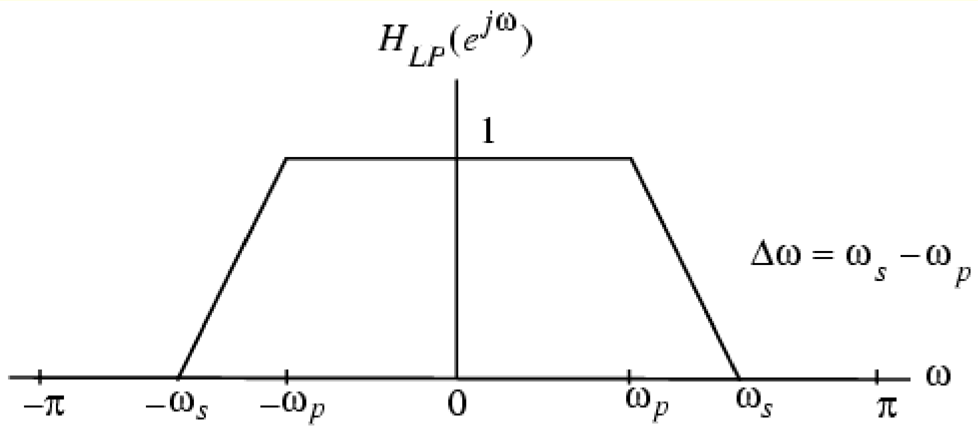
where $w[n]$ is the n-th coefficient of a length-25 Kaiser window with $\beta = 3.3953$





Impulse Responses of FIR Filters with a Smooth Transition

- First-order spline passband-to-stopband transition



$$\omega_c = (\omega_p + \omega_s)/2$$

$$\Delta\omega = \omega_s - \omega_p$$

$$h_{LP}[n] = \begin{cases} \omega_c/\pi, & n = 0 \\ \dfrac{2\sin(\Delta\omega n/2)}{\Delta\omega n} \cdot \dfrac{\sin(\omega_c n)}{\pi n} & |n| > 0 \end{cases}$$

Impulse Responses of FIR Filters with a Smooth Transition

- Pth-order spline passband-to-stopband transition

$$h_{LP}[n] = \begin{cases} \omega_c / \pi, & n = 0 \\ \left(\dfrac{2\sin(\Delta\omega n / 2P)}{\Delta\omega n / 2P} \right)^P \cdot \dfrac{\sin(\omega_c n)}{\pi n} & |n| > 0 \end{cases}$$

Lowpass FIR Filter Design Example

- Example