Errata, Hints, and Problem Solutions

to accompany the text

Probability, Random Variables, and Stochastic Processes, Fourth Edition,
by Athanasios Papoulis and S. Unnikrishna Pillai,
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Objective

This volume was prepared by the instructor for students of Northeastern University course ECE 3211, Applied Probability and Stochastic Processes, to help in dealing with the course Text. As the title indicates, there are three parts: Text errata, followed by hints for solving the problems, followed by problem solutions. The course does not cover the entire Text; nor do these notes. Chapters 1 through 7, Chapters 9 through 11, and the first Section of Chapter 12 are covered here. The course does not cover all the topics in Chapters 10 and 11, but all the problems in those chapters are included here.

The main part of this work is the problem solutions. The glory of the Text is its problems. Many of them contain key parts of important developments in the field, and are widely usable. Doing the problems is where real learning happens. There are so many problems and some are so difficult as to daunt all but the most intrepid student. I cannot read and correct homework, and I am often not available when help is needed, especially in getting started, so this volume is an attempt to fill the void.

Reading the solution to a problem without first seriously attempting to solve it yourself is a mistake. You may well learn something, but it will not have the staying power of an answer you developed. There are exceptions. Some of my students do not have enough training in mathematics to approach a few of the more mathematically oriented problems. The hints section identifies those problems that (I think) are important or difficult, as well as those few that have typographical errors, missing background material, incomplete assumptions, or indemonstrable results.

The attempt here is not to provide the typical "answer book." Rather, it is an attempt to turn each problem into a worked example so that the student may follow the solution and see it flow from the body of the Text. As an instructor, I have access to the publisher-provided solutions manuals. Following my own advice, I have not used those manuals in developing the solutions here. That means, sometimes, that an easier road to the solution of a particular problem might well exist in the official manual, although those are not generally available to students.

For the Second and Third Editions of the Text, this volume was reproduced from hand-written masters. For the Fourth Edition, it has been newly typeset. Typographical (and other) errors are likely to exist, and are especially confusing to students. I would be grateful for any corrections or comments, which might be addressed to

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Note that this volume is split into three parts for electronic dissemination as PDF files. The first part contains just the errata and the hints. The second part has the solutions for Chapters 2 through 6, and the third part has the solutions for Chapters 7, 9, 10, 11, and the first section of Chapter 12.

ERRATA & COMMENTS ON THE TEXT

(Fourth Edition)

The Fourth Edition of the Text is (almost) new at this writing. The errata here are for the first printing. Errors are sometimes corrected with new printings. The printing number is the first integer remaining in the sequence 1234567890 on the reverse of the title page, above the ISBN line. Since this Text is newly revised, I have not thoroughly examined it, and have likely overlooked some errors. I would appreciate notification of any errors not listed here (or any comments on errors listed here) to GMatchett@northropgrumman.com.

CHAPTER 2

- p. 19, first paragraph of Section 2-2, line 2, should read "... and certain of its subsets events."
- p. 19, first paragraph of Section 2-2, line 4, should read "... ζ_i is an elementary event, if, in fact, $\{\zeta_i\}$ is an event at all."
- p. 29, following Eq. (2-38), should read "...results involving probabilities hold also...".
- p. 32, following Eq. (2-42), note that "This result" refers to Eq. (2-41), and not to Eq. (2-42).

CHAPTER 3

- p. 53, Fig. 3-2 (a) should read $\frac{2}{3\sqrt{2\pi}}e^{-(x-4.5)^2/4.5}$.
- p. 53, Fig. 3-2 (b) should read $\frac{1}{\sqrt{4\pi}}e^{-(x-3.0)^2/4}$.
- p. 71, Problem 3-7 should read "... net gain or loss exceeds...".
- p. 71, Problem 3-5 should read "We pick at random $n \le N$ components ...".

CHAPTER 4

- p. 79, 3. Proof, should read "Suppose that $\mathbf{x}(\zeta) > 0$ for every ζ."
- p. 91, Eq. (4-48) should read $0 < x < 1$.
- p. 91, Eq. (4-49), the upper limit on the second integral should be $\pi/2$.
- p. 96, Watch out! There are two different definitions for both the negative binomial distribution and for the geometric distribution. When these distributions are specified, as in the problems, it is not always clear which of the two is meant.
- p. 99, line 4 (in Example 4-14) should read $20 \le x < 40$.
- p. 100, lines 5 and 6 should read $b < x \le a$.
- p. 100, line 10 should read $b < x < a$.
- p. 102, Example 4-17, the equation for the density is incorrect. The G symbols should

be $\boldsymbol{g}$ symbols (which were dropped from the Fourth Edition). Note that

$$\boldsymbol{g}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad \text{so that} \quad \boldsymbol{G}(x) = \int_{-\infty}^{x} \boldsymbol{g}(\xi)d\xi$$

(compare with Eq. (4-27), and see Notational Peculiarities of the Hints section).

- p. 106, Table 4-1, note that the Text definition of $\mathrm{erf}(x)$ is nonstandard. The relationship is

$$\mathrm{erf}(x) = \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\Phi\left(\frac{x}{\sqrt{2}}\right)$$

where the first term is the Text definition, the second term is the standard definition, and the third term is the "Gradshteyn & Ryzhik" notation.

- p. 107, first line after Eq. (4-93), the restriction $x > 0$ is not needed.
- p. 119, Problem 4-3 should read "Using Table 4-1 ...", and, in part (b), "$\boldsymbol{x}$ is $N(\eta, \sigma^2)$".
- p. 120, Problem 4-8 should read"$\boldsymbol{x}$ is $N(10, 1)$".
- p. 120, Problem 4-10 should read "$\boldsymbol{x}$ is $N(0, 4)$".
- p. 120, Problem 4-12 should read "$\boldsymbol{x}$ is $N(1000, 400)$".
- p. 120, Problem 4-14, the reference to Eq. (4-34) is not correct. The actual equation intended was removed in going to the Fourth Edition. Eq. (4-90) is a close equivalent to what was intended.
- p. 120, Problem 4-15 should read "... then $F(x) = 1$ for $x \ge b$ ...".
- p. 121, Problem 4-26 should read "A system has 1000 components ."
- p. 121, Problem 4-28, the last ">" should be a "<" in the Hint.
- p. 122, Problem 4-35 should say in addition "Assume also: $k_1, k_2 \ll n, k_3$".

CHAPTER 5

- p. 133, equation after Eq. (5-24), the last term should be $U(x)$.
- p. 145, Example 5-22, next to last line, should read "... that $E\{(\boldsymbol{x} - \eta)^2\} = \sigma^2$ and ...".
- p. 151, equation following Eq. (5-88), should read

$$P\{|\boldsymbol{x} - \eta| \ge \varepsilon\} = \int_{-\infty}^{\eta - \varepsilon} f(x)dx + \dots$$

- p. 153, Example 5-28, should read "... function of an $N(\eta, \sigma^2)$ random variable ...".
- p. 155, two lines after Eq. (5-111), the right hand equation should have a double prime on the phi on the left hand side.
- p. 164, Problem 5-1, should read $N(5, 4)$.
- p. 165, Problem 5-17, should read $\boldsymbol{y} = \sqrt{\boldsymbol{x}}$.
- p. 166, Problem 5-38, (a) should read $\Phi_x(\omega) = (1 - j\omega\beta)^{-\alpha}$, and (b) should read

  $\Phi_x(\omega) = (1 - j2\omega)^{-n/2}$.
- p. 168, Problem 5-51 (b), the equation line should end $k \le \min(M, n)$.

CHAPTER 6

- p. 182, Eq. (6-43), lower limit on the first integral should read $y = -\infty$.
- p. 208, Eq. (6-157), see Special Note 3 in the Hints section of this volume.
- p. 219, Example 6-36, line 5, should begin "$\phi(s_1, s_2) = e^A$".
- p. 236, Problem 6-8, second centered equation, left side should read $f_z(z) =$.
- p. 237, Problem 6-25, see Special Note 2 in the Hints section of this volume It is easiest if it reads "... exceeds $2/\lambda$.", and "... original component by $1/\lambda$?"
- p. 238, Problem 6-36, should ask to show that w is an exponential random variable.
- p. 239, Problem 6-43 has "excess" information. See the hint for this problem.
- p. 241, Problem 6-67, the first display equation should read

$$E\{\mathbf{z}\} = \sum_n p_n E\{g(x_n, \mathbf{y}) | x_n\}$$

- p. 242, Problem 6-76, should read "..., $\beta_y(t) = f_y(t | (\mathbf{y} > t))$ and ...".

CHAPTER 7

- p. 243, Eq. (7-3), should not have commas in the denominator of the fraction.
- p. 260, line 17, should read "as in Example 7-5. ..."
- p. 276, second line of the section "Ergodicity" should refer to Sec. 12-1.
- p. 279, Example 7-15, should read "... in the interval $(0, T)$."
- p. 283, Eq. (7-136), should read

$$\int_{-\infty}^{\infty} |x|^{\alpha} f_i(x) dx < K < \infty \quad \text{for all } i$$

- p. 283, Eq. (7-138), should read

$$E\{|\mathbf{x}_i|^3\} \le c\sigma_i^2 \quad \text{all } i$$

- p. 291, Example 7-21, in two places the reference to (7-15) should be to (7-156).
- p. 302, Problem 7-32, should begin "... are normal, uncorrelated with zero mean ...".

CHAPTER 9

- p. 378, Example 9-5, results are valid for positive times only. The right hand term of the equation following Eq. (9-14) is not correct if $t_1 = t_2 = t$, where it produces $2\lambda t$. Correctly, $C(t, t) = \lambda t$.
- p. 379 & 380, equations following Eq. (9-18) are only true if $t_2 \le t_1$. In case $t_1 \le t_2$, all the t_2's in these equations should be replaced by t_1's.
- p. 384, Eq. 9-37, should read $E\{|\mathbf{s} - \eta_s|^2\} = \int_a^b \dots$
- p. 391, first line of "Proof" near page bottom should read "...[See (6-242)]"
- p. 397, Example 9-17 should assume a SSS process.
- p. 397, equation after Eq. (9-83) should have $g(x) = +c$ when $x > c$, and $g(x) = -c$

when $x < -c$.

- p. 399, Eqs. (9-92) and (9-93) are only true for a real system. When the system is complex, the conjugates L_2^* and h^* should appear.
- p. 400, equation before Example 9-18 should read "…$h(\alpha)h^*(\beta)$…".
- p. 404, Eq. (9-111) should begin "$a_n\mathbf{y}^{(n)}(t) + \ldots$".
- p. 412, next to last line, should read "… cases of (9-92) and (9-93)." Note that $H(\omega)$ is used here without any introduction. It is the *system function*, where

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt \qquad h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega$$

- p. 412, Eq. (9-150) uses $\rho(\tau)$ without calling it the *deterministic autocorrelation* of $h(t)$, and the next line (on p. 413) should read " … white noise with intensity q.
- p. 413, two lines before Eq. (9-155) should read "… It is thus a simple low-pass filter."
- p. 414, Eq. (9-157) should read

$$S_{yy}(\omega) = |j\omega|^{2n}S_{xx}(\omega) \qquad R_{yy}(\tau) = (-1)^n R_{xx}^{(2n)}(\tau)$$

- p. 415, Example 9-28, line 3, should read "… equals $\frac{1}{\pi t}$".
- p. 419, Eq. (9-179) should read "$E\{|\mathbf{x}(t+\tau_1) - \mathbf{x}(t)|^2\} = 0$".
- p. 412, Eq. (9-194), the sum should begin at $m = 1$.
- p. 425, Example 9-34, note that a is presumed real, and line 5 should read

$$R_{yy}[m] = \frac{q}{1-a^2}a^{|m|}$$

- p. 430, Problem 9-6, should read

$$E\{\mathbf{w}^2(t)\} = \int_0^t (t-\tau)^2 q(\tau)d\tau$$

- p. 433, Problem 9-41, should read

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi}S_x(\omega)*S_x(\omega)$$

- p. 433, Problem 9-51, should read "$R[0]R[2] \geq 2R^2[1] - R^2[0]$"
- p. 433, Problem 9-52, should read "$\mathbf{x}[n] = Ae^{jn\omega}$." ($\omega$ is a random variable here; I cannot make bold greek symbols.)

CHAPTER 10

- p. 447, line after Eq. (10-58) should read "$k = 1.37 \times 10^{-23}$ Joules/degree-K".
- p. 450, first word, should read "inertia has the…".
- p. 461, four lines after Eq. (10-103), should read "… and $E\{\mathbf{n}^2(t)\} = \lambda^2 t^2 + \lambda t$."
- p. 466, Eq. (10-142) (second part) should read "$R_{x\hat{x}}(-\tau) = -R_{x\hat{x}}(\tau)$".
- p. 469, line three, should read "… that $S_{zz}(\omega)$ is specified. …".

- p. 473, two lines before Eq. (10-173), should read "SSCS process with period T and ...
- p. 474, Eq. (10-180), should read

$$\bar{S}_x(\omega) = \frac{1}{T}\mathsf{S}_c(e^{j\omega T})|H(\omega)|^2 \text{ ''}$$

- p. 474, Eq. (10-182) is improper. To fix this, define $\boldsymbol{w}(t) = \sum_{n=0}^{\infty} \boldsymbol{c}_n U(t - nT)$ for $t \geq 0$, with an appropriate alternate definition for $t < 0$. Now $\boldsymbol{w}(0^-) = 0$.
- p. 474, last equation should have the term $R_c[n - r]$, not $R_c(n - r)$.
- p. 475, Eq. (10-183), should read

$$\ldots = \sum_{m=-\infty}^{\infty} R_c[m] \sum_{r=-\infty}^{\infty} \delta[t + \tau - (m + r)T]\delta(t - rT)$$

- p. 475, equation after Eq. (10-183) should read

$$\sum_{r=-\infty}^{\infty} \int_0^T \delta[t + \tau - (m + r)T]\delta(t - rT)dt = \delta(\tau - mT)$$

- p. 475, Eq. (10-185) should read

$$\bar{S}_z(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} R_c[m]e^{-jm\omega T} = \frac{1}{T}S_c(\omega T)$$

- p. 476, Eq. (10-189) should indicate equality in mean square.
- p. 477, Eq. (10-194) presumes the process is real.
- p. 480, five lines before Eq. (10-203), should read "$T_0 \leq \pi/\sigma$ ...".
- p. 493, line before Eq. (10A-1) should read "... for any $c > 0$".
- p. 494, line 3 is incorrect. See the solution to Problem 10-23.
- p. 496, Problem 10-13, equation should read "$S_x(\omega) = \frac{2\pi}{T^2}\left|\int_0^T \ldots\right.$ "
- p. 497, Problem 10-21, needs the assumption that $\boldsymbol{x}(t)$ is independent of all $\boldsymbol{t}_i$, and should read "$\boldsymbol{X}_c(\omega) = \frac{1}{\lambda} \sum_{|\boldsymbol{t}_i| < c} \boldsymbol{x}(\boldsymbol{t}_i)e^{-j\omega \boldsymbol{t}_i}$". It is not needed that $E\{\boldsymbol{x}(t)\} = 0$.
- p. 497, Problem 10-25, should read "$\boldsymbol{y}(t) = B\cos(\omega_0 t + \varphi) + \boldsymbol{y}_n(t)$".

CHAPTER 11

- p. 506, two lines after Eq. (12-30), should read "... system of Fig. 11-4 is ...".
- p. 507, line after Eq, (11-37), should read "$\alpha_i = \gamma_i \mathsf{L}(1/z_i)$". (I cannot duplicate the Text fonts, but there is a problem of consistency here.)
- p. 508, line before Eq. (11-44), should refer to Example 9-32.

- p. 515, line before Eq. (11-78), should read "$\mathbf{B}(-\omega) = -\mathbf{B}(\omega)$".
- p. 515, line after Eq. (11-79a) should refer to Eq. (11-70), not (11-9).
- p. 521, Problem 11-7, should read

$$\beta_n = \left(a + \frac{\lambda_n}{2}\right)^{-1/2} \qquad \beta_n' = \left(a + \frac{\lambda_n'}{2}\right)^{-1/2}$$

- p. 522, Problem 11-10, should read "$E\{\boldsymbol{x}_n \boldsymbol{x}_k^*\} = \ldots$".

CHAPTER 12

- p. 527 Eq. (12-9), should read "$\int_0^T |C(\tau)| d\tau < \infty$".

PROBLEM HINTS AND COMMENTS

1) Introduction

Several symbols are used to comment on the problems, often subjectively. They are

- **I**, an important problem, likely one whose results will be needed later
- **M**, a moderately difficult problem
- **D**, a difficult problem
- **F**, a flawed problem, generally containing a typographical error, a missing assumption, or an indemonstrable result
- **B**, a problem presuming background information not presented in the Text up to the point of the problem, or not presented at all

Notational Peculiarities

Where the Text uses $\{\varnothing\}$ for the null, or empty, set, we use $\varnothing$.

The symbol used here for the Gaussian distribution function, defined in Eq. 4-27 of the Text, is $\mathbb{G}(x)$.

The Fourth Edition of the Text does not use the symbol $\boldsymbol{g}(x)$ (except in Problem 4-28), used in the Third Edition, and occasionally here, for the Gaussian density function. The Fourth Edition thinks that no symbol is necessary for this function, since

$$\boldsymbol{g}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The notation $\ln x$ means the natural logarithm of x. The Text sometimes uses $\log x$.

Special Notes

1. The Text changed notation in going from the Third Edition to the Fourth Edition. In the Third edition, a normal random variable with mean η and standard deviation σ was denoted as $N(\eta;\ \sigma)$, which was sloppily written as $N(\eta,\ \sigma)$ from time to time. In the Fourth Edition, the official notation is $N(\eta,\ \sigma^2)$. Unfortunately, not all instances of the older notation were changed. This is particularly troubling when actual numbers are used for the parameters, as they are in some problems. Does the notation $N(100,\ 25)$ mean a normal random variable with a standard

deviation of 25 or of 5? One cannot be sure. These hints will provide my guess, from comparison of the two editions, and from a reading of the official Solutions Manual.

2. Eq. 4-30, p. 85, defines the distribution of an exponential random variable with parameter λ to have the p.d.f. $f_x(x) = \lambda e^{-\lambda x} U(x)$. The parameter has units that are the inverse of the units of the random variable itself (often $\boldsymbol{x}$ has units of time, and λ has units of (1/time)). It makes, perhaps, more sense to use the inverse of λ as the parameter, so that the parameter and the variable share the same units, and the p.d.f. is $f_x(x) = (1/\lambda) e^{-x/\lambda} U(x)$, but that was not done, except in some of the problems. When a problem states that $\boldsymbol{x}$ is an exponential random variable with parameter λ, one cannot be sure which of the two possibilities is meant, except by implication. For example, if it asks: What is the probability that $\boldsymbol{x}$ exceeds 2λ? (see Problem 6-25), then it must be that $\boldsymbol{x}$ and λ have the same units, so the inverse parameter is indicated.

3. Eqs. 4-49 and 4-51 define and evaluate the beta function, which is denoted as $B(\alpha, \beta)$. In some places (e.g., Eq. 6-157) this function is denoted as $\beta(m, n)$. For further confusion, the binomial distribution is sometimes denoted as $B(n, p)$ (see Theorem 5-2 or Example 5-30).

References

The problems are more easily done with three tools: a scientific calculator, a set of good function tables, and a table of integrals and sums. I use and recommend:

AS — Milton Abramowitz and Irene A. Stegun, Editors, *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series 55, June 1964 (now available as a Dover paperback).

GR — I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products*, Fourth Edition, Academic Press, 1965 (Sixth Edition, 2000, now available).

A excellent book for general reference here is:

FI — William Feller, *An Introduction to Probability Theory and Its Applications*, Volume I, Second Edition, Wiley & Sons, 1957 (Third Edition, 1990, is current).

2) Chapter 2 (there are no problems in Chapter 1)

2-1 Begin with DeMorgan's laws, Eq. 2-5.

2-2 No hint.

2-3 Try to find some set C such that $\bar{B} = A + C$, and $AC = \varnothing$. Then apply Eq. 2-10.

2-4 (a) Use Eq. 2-10. (b) Use Eqs. 2-12 and 2-13.

2-5 Use Eq. 2-13, repeatedly.

2-6 M, B Use the fact that a set is countable if it is empty or is the range of some sequence. Show that any subset of a countable set is countable. Then use the countable union property of Borel fields to show that every subset of S is an event.

2-7 List all subsets of S. Beginning with the list $\varnothing$, S, $\{1\}$, and $\{2, 3\}$, form compliments and unions among the list items to find new subsets that must be in the Borel field and add them to the list. This process stops when nothing new can be found.
Note that any finite Borel field must have 2^n elements for some integer n.

2-8 Use the definition of conditional probability, Eq. 2-33.

2-9 No hint.

2-10 I No hint.

2-11 M See the solution for help.

2-12 Presume classical probability theory with the probability of an interval of points being proportional to its length.

2-13 D, B For an easier problem, assume that $P\{t \le t_1\} = F(t_1)$ is a continuous, differentiable function of t_1, and assume $F(0) = 0$.

2-14 I No hint.

2-15 I, M Enrich this problem by letting B_i be any of the sets A_i, $\overline{A_i}$, S, or $\varnothing$.

2-16 M, B Problems 2-16 through 2-19 and 2-21 are problems in combinatorics that the Text has not yet considered. It would be good to at least solve Problem 2-26 first, to begin the topic. Let an outcome of the experiment here be a k-element sequence of distinct numbers selected from the set 1 to n. Find out how many outcomes there are. (a) Next, find out how many outcomes there are that contain no number larger than m. Call this result M_m. Then notice that the number of outcomes that have m as the largest number is $N_m = M_m - M_{m-1}$. (b) The number of outcomes with a largest number less than or equal to m is the number of outcomes with no number larger than m.

2-17 B Work Problem 2-16 first. The difference here is that the outcomes are k-element sequences of numbers that are not necessarily distinct.

2-18 B Work Problem 2-26 first.

2-19 B Number the black balls from 1 to n, and number the white balls from $n+1$ to $n+m$. Now the problem is: what is the probability that, if k balls are drawn, the highest numbered will be $n+1$ or more, and Problem 2-16 will be useful.

2-20 Consider the outcome of the experiment to be the point where the center of the penny lands, and consider that events are sets of points with areas, and that probability is proportional to area.

2-21 B (a) This can be done by appealing to Problem 2-16 again. (b) Be sure you understand Problem 2-26.

2-22 Find out how many subsets with two or more elements there are of a set of n elements. Relate each of these subsets to an equation needed for independence.

2-23 Use Bayes' theorem, Eq. 2-44.

2-24 Use total probability, Eq. 2-41.

2-25 Draw a diagram, something like Fig. 2-12c. Note that the area of the diagonal strip equals the area of the square less the area of the corner triangles.

2-26 I, M First, count the different <u>sequences</u> of k distinct elements taken from a set of n elements. Then, consider how many different <u>subsets</u> of k distinct elements taken from a set of n elements there are.

2-27 Use Bayes' theorem, Eq. 2-44.

3) Chapter 3

3-1 A occurs two or more times if it does not occur zero or one time.

3-2 A simple application of Problem 3-1b.

3-3 Find the probability that seven will not show at all.

3-4 Write down the binomial theorem expansions of $(q+p)^n$ and $(q-p)^n$, then add them together.

3-5 Deduce that the number of ways to take n items from N items, so that a subset of k of the n items come from a subset of K of the N items, is the product of the number of ways to take k items from K items with the number of ways to take $n - k$ items from $N - K$ items.

3-6 Apply Problem 3-1.

3-7 First, find out how many (what range of) wins are needed to bound the amount won or lost to the amounts specified, then compute the probability of having that many wins. The unnumbered equation on p. 57, just after the “**Proof.**” heading, will be useful in speeding up the computations, if you are doing them on a hand calculator.

3-8 Deduce that having r successes in all n trials, including a success on the ith trial, is the same thing as having $r - 1$ successes in the $n - 1$ trials that exclude the ith trial, along with a success on the ith trial.

3-9 **F** The problem is not well stated. Does it ask for the probability that any one (or more) of the four players will have all 13 cards of any one suit? Restate the problem so that it asks what is the probability that a specified one of the players, will have a perfect hand.

3-10 **D**, **B** What does the “average duration” of such a game mean? As yet, it has no meaning. What is meant is the expected value of the total number of games. The expected value is a concept introduced in Section 5-3. You might put this problem off until the concept of an expected value, and that of a conditional expected value, are clear. Define $\boldsymbol{n}_k$ to be the random variable that is the number of games until either A or B is ruined, supposing that A starts with capital k. Deduce the total probability theorem for random variables, which in this case is (see Problem 5-27)

$$N_k = E\{\boldsymbol{n}_k\} = E\{\boldsymbol{n}_k | H\}P(H) + E\{\boldsymbol{n}_k | \overline{H}\}P(\overline{H})$$

Let H be the event that A succeeds in the next game. This will establish the hint given in the problem. Solving the difference equation of the hint is somewhat difficult. Define the increments $\Delta N_k = N_k - N_{k-1}$, find the difference equation for the increments, and note that $\Delta N_1 = N_1$, where N_1 is temporarily unknown. Use the solution for the increments (in terms of the unknown N_1) to solve for the average durations themselves, then use the fact that $N_{a+b} = 0$ to find N_1 and all the other average durations.

3-11 **D**, **F** The concept of stakes may not be clear. What is meant is that, on each play, A

bets an amount α, while B bets an amount β, so that A wins β, or he loses α. What remains unclear is the concept of ruin in this game. Is A ruined when he has no money left? Or, is A ruined when he has less than his stake, α, left so that he may no longer play? The latter is a more reasonable definition. Even with this clarification, the problem is too difficult to work in general. Try a specific example: let $\alpha = 2$, $\beta = 3$, and $a + b = 8$. Then change to $a + b = 9$ and see how the problem changes character.

3-12 B Again, the premature use of the term "expected," a term not properly used until Section 5-3. Here, the expected loss is the sum of the various possible losses, each multiplied by its probability.

4) Chapter 4

4-1 I, **F** This problem cannot be done unless it is assumed that $F(x)$ is invertible. See the solution for a lengthy discussion.

4-2 F Comments here are similar to those for Problem 4-1.

4-3 F (a) A typo: the reference should be to Table 4-1. This problem points up the need for better tables than Table 4-1 of the Text. The reference AS (Tables 26.5 and 26.6) is a good source. Use Eq. 4-26. (b) Another typo. See Special Note 1.

4-4 The same remarks apply here as for Problem 4-3.

4-5 No hint.

4-6 Use Eq. 4-8.

4-7 See p. 88 for the Erlang random variable.

4-8 Follow Example 4-15.

4-9 Note that $\frac{d}{dx}U(x-c) = \delta(x-c)$.

4-10 F See Special Note 1. The problem should read $\boldsymbol{x} \sim N(0, 4)$. (b) Use Eq. 2-33.

4-11 I Show $\{t|\boldsymbol{x}(t) \le x\} = \{t|t \le G(x)\}$ for G increasing.

4-12 F See Special Note 1. The problem should read $\boldsymbol{x} \sim N(1000, 400)$. (a) Use Eq. 4-1. (b) Use Eq. 4-67.

4-13 Equations which were in the Third Edition of the Text, but have been removed, are for the density and distribution function of the binomial random variable. They are

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(x-k)$$

$$F(x) = \sum_{k=0}^{m} \binom{n}{k} p^k q^{n-k}, \quad \text{where } m \le x < m+1 \text{ defines } m\text{, given } x.$$

4-14 **F** There is a typo here. The reference is to old Eq. 4-34, which no longer exists. It said for a binomial random variable that $F(x) \approx \mathbb{G}\left(\frac{x-np}{\sqrt{npq}}\right)$. This may be deduced from (new) Eq. 4-90. Use the equation for $f(x)$ in the hint for 4-13 above for the exact result.

4-15 No hint.

4-16 **I** Use set theory reasoning.

4-17 **B** This problem is seriously out of place. You need Eq. 6-224 to see how the conditional failure rate function is related to the distribution function for the nonnegative random variable $\boldsymbol{x}$. Then use the Rayleigh density from Eq. 4-44.

4-18 Use Eq. 2-41 (total probability) with $M = \{\zeta | \boldsymbol{x}(\zeta) \le x\}$ and $\overline{M}$.

4-19 Use Eqs. 4-67 and 2-33.

4-20 **M** Use Eq. 4-80 with $M = A \cap \{\zeta | \boldsymbol{x}(\zeta) \le x_0\}$.

4-21 Use Eq. 4-80, noting that $\int_0^1 p^j (1-p)^k dp = \frac{j!k!}{(j+k+1)!}$.

4-22 Do Problem 4-21 first. Use Eq. 4-82. Leave your answer in integral form. For a numerical evaluation, see the solution.

4-23 Use Eq. 4-96.

4-24 Use Eq. 4-96.

4-25 Use Eq. 4-96. For large x, approximate $\mathbb{G}(x) \approx 1 - \frac{1}{x}\boldsymbol{g}(x)$ from Problem 4-28.

4-26 **F**, **M** First fix the typo: The system should have 1000 components, not 100. Next, note that Eq. 4-100 does not apply. Use Eq. 4-89 and Stirling's formula, Eq. 4-111, for numerical results.

4-27 Deduce that a head must have come up at the nth flip, so that $k-1$ heads must have come up in the first $n-1$ flips.

4-28 F Note the typo: the second inequality of the hint should contain a "less than" sign rather than a "greater than" sign. No hint.

4-29 Find the probability that A does not happen in n trials.

4-30 Use Eq. 4-107. To add insight, consider a slightly different problem, as well. Consider that accidents are a Poisson point process, with an average rate of 0.02 accidents per month, and ask then what is the exact probability that a driver will have 3 accidents in 100 months.

4-31 Use Eq. 4-102.

4-32 No hint.

4-33 I Deduce that if eleven does not show on the first roll, then the probability that Y will win is the same as the original (before the first roll) probability that X will win.

4-34 I (b) **M** the solution.

4-35 F, M It is necessary to assume $k_1 \ll k_3$ and $k_2 \ll k_3$.

4-36 Use Eq. 4-115 and 4-116.

5) Chapter 5

5-1 F See Special Note 1. Consider this a typo. The problem should say that $\mathbf{x} \sim N(5, 4)$. Use Eq. 5-18.

5-2 Use Eq. 5-18, Examples 4-12 and 5-1b.

5-3 See Example 5-3.

5-4 Use Eqs. 5-22 and 4-16.

5-5 Follow Example 5-4 to find $F_y(y)$. Differentiate to find $f_y(y)$, being careful to consider discontinuities in the distribution function.

5-6 Use the "fundamental theorem," Eq. 5-16.

5-7 Begin with Eq. 4-6, then use Eqs. 4-116 and 4-117.

5-8 Use Eqs. 5-16, 4-30, and 4-44.

5-9 Use Eq. 5-16, but remember that $F_x(x)$ may not be continuous. In part (a), there could be a discontinuity at $x = 0$. In part (b), the value $\boldsymbol{y} = 0$ corresponds to an entire range of $\boldsymbol{x}$ values.

5-10 (a) The value $\boldsymbol{y} = 0$ corresponds to an entire range of $\boldsymbol{x}$ values.

5-11 **F** Assume that the symmetry point of the Cauchy density—the value μ in Eq. 4-52—is zero. The symbol μ is often used for the mean value of a random variable, but a Cauchy random variable has no mean value.

5-12 No hint.

5-13 Presume a function $\boldsymbol{y} = y(\boldsymbol{x})$, where $y = y(x)$ has one root in the range $x \in (-1, 1)$ for all $y \geq 0$. Find a differential equation for $y(x)$ and solve it.

5-14 No hint.

5-15 (a) See the hint for Problem 4-13. (b) Some (individual) values of $\boldsymbol{y}$ correspond to more than one value of $\boldsymbol{x}$.

5-16 The beta distribution is defined on p. 91. Note the typo in Eq. 4-48: the symbol b should be 1.

5-17 First, correct the typo. It should be that $\boldsymbol{y} = \sqrt{\boldsymbol{x}}$. Use Eqs. 4-39 and 5-16.

5-18 Use Eqs. 5-16 and 4-39.

5-19 Use Eqs. 5-16, 4-30, and 4-43.

5-20 **D** Try an easier problem: Assume $f_x(t)$ is continuous at $t = 0$. Use Illustration 7, on page 134.

5-21 Use the development on p. 100 of the Text.

5-22 No hint.

5-23 Use Eq. 5-76.

5-24 Use Eq. 5-73.

5-25 **I** Use Eq. 4-56 for the binomial random variable. (a) Use Eq. 5-46 for a direct approach, which is somewhat tricky. Make use of the binomial theorem,

$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$, good for all integer n and all p and q. Differentiate the theorem equation with respect to the variable p. An easier approach is to take advantage of Example 5-30, Eq. 5-117, and the moment theorem, Eq. 5-115.

5-26 (a) Use the Chebyshev inequality, Eq. 5-88. (b) Use the moment theorem, Eq. 5-115.

5-27 I Use Eqs. 4-74, 5-44, and 5-47.

5-28 Use the Markov inequality, Eq. 5-89.

5-29 No hint.

5-30 No hint.

5-31 No hint.

5-32 M (a) Use $|x-m| = \begin{cases} (a-x)+(m-a) & x \le m \\ (x-a)-(m-a) & x \ge m \end{cases}$.

(b) Deduce that $\int_a^m (x-a)f(x)dx \ge 0$ for any a.

5-33 M Write $E\{|\mathbf{x}|\} = \frac{1}{\sigma\sqrt{2\pi}}\left\{\int_0^{\infty} xe^{-(x-\eta)^2/(2\sigma^2)} - \int_{-\infty}^{0} xe^{-(x-\eta)^2/(2\sigma^2)}\right\}$. Then substitute $z = (x-\eta)/\sigma$ in both integrals. Explicitly integrate what you can and relate the rest to $\mathbb{G}(\eta/\sigma)$.

5-34 M Use the basic inequality $\ln z \le z-1$.

5-35 No hint.

5-36 See the Lyapunov inequality, p. 152.

5-37 M (a) Either assume that $\mu = 0$ in Eq. 4-52 defining the Cauchy density, or, better, deduce a slightly different result than the problem asks. Use Cauchy's residue theorem for contour integration to find the value of the integral. Or, just use a good table of integrals, such as reference GR.

5-38 This is a workhorse problem. (a) **F** Note the typo: The characteristic function should read $\Phi(\omega) = (1-j\beta\omega)^{-\alpha}$. (See Table 5-2.) (c) See Problem 5-25. (d) **F** The section on the negative binomial random variable is confusing. Two different distributions share this name. One is described by Eq. 4-62 or 4-63, and the other is described by Eq. 4-64. This duality is reinforced by Table 5-2. Comparing the Table with the answer wanted in this problem leads to the conclusion that Eq. 4-64 should be chosen here. While the problem does not request it, it is illustrative to find $E\{\boldsymbol{x}\}$, to see how sums may be manipulated in the same ways as integrals.

5-39 Use Eqs. 5-113 and 5-116.

5-40 Note that $(1-y)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}(-y)^k = \sum_{k=0}^{\infty}\binom{n+k-1}{k}y^k$. See the solution for more on binomial coefficients.

5-41 M It is clear from the problem that we are discussing the negative binomial random variable of Eq. 4-63. Techniques developed for Problem 4-35 will be useful here.

5-42 Note that $e^{s(\boldsymbol{x}-\eta)} = \sum_{k=0}^{\infty}\frac{s^k(\boldsymbol{x}-\eta)^k}{k!}$.

5-43 Split the integral in the hint in the Text into real and imaginary parts.

5-44 Extend the development following Eq. 5-111 to higher derivatives.

5-45 No hint.

5-46 No hint.

5-47 I, M Consider, for small ε, $E\{g(\boldsymbol{x}-\eta-\varepsilon)\}$, by expanding the function $f(\eta+x+\varepsilon)$ in a Taylor series in ε about the point $\eta+x$. Use the obvious symmetry and maximum and limiting properties of $g(x)$ and $f(\eta+x)$ that are deduced from the graphs.

5-48 (a) **D** Regard the density of $\boldsymbol{x}$ as a function of both x and v. Show that $\frac{\partial^2 f(x,v)}{\partial x^2} = 2\frac{\partial f(x,v)}{\partial v}$. Integration by parts will then prove the theorem. (b) Use part (a) with $g(\boldsymbol{x}) = \boldsymbol{x}^n$.

5-49 I Use the fundamental theorem of Fourier series.

5-50 F The description of the experiment is confusing. What is the length of the run? The problem assumes that the first tossing is not included in the run, so that the run may end on the tossing following the first tossing, in which case $\boldsymbol{x} = 1$, not 2. Note also that the p.m.f. refers to the moment function?

5-51 How is it possible that two items are identical, yet one is defective and one is not? Any difference makes them not identical. Perhaps they are merely similar. Forget this cavil. (b) **F, D** There is a typo: the equation line should end with $\min(M, n)$ and not with $\min(M, N)$. See Table 5-2 for the answers here. To compute the expected value of $\boldsymbol{x}$ you will need Vandermonde's identity, which is

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$$

The computation of $\mathrm{Var}(\boldsymbol{x})$ is even more difficult; a reference is provided.

(c) Notice that, as $M, N \to \infty$ while $M/N = p$ is fixed, $\dfrac{M!}{(M-k)!}\dfrac{(N-k)!}{N!} \to \left(\dfrac{M}{N}\right)^k = p^k$

5-52 (a) To get the r'th white ball on the k'th draw implies that $r-1$ white balls are drawn in the first $k-1$ draws, and a white ball is drawn on the k'th draw.

(b) **D** Consider the number of ways of ordering all the balls. Find out how many of these ways are favorable to the desired outcome.

(c) Find the limit of the result in part (b) in a manner similar to that used in Problem 5-51c.

6) Chapter 6

6-1 A workhorse problem, but parts have already been done in examples. (a) Use Eq. 6-45. (b) Use Eq. 6-54. (c) **D** The Text omits the prime example where $\boldsymbol{z} = \boldsymbol{xy}$. You have three options: 1) Work out this case generically, using Example 6-10 as a point of departure. 2) Work out this case for positive random variables only, where it simplifies somewhat. 3) Skip ahead to Eq. 6-148. Once the generic result is established, the specific result will be an integral difficult to evaluate. Leave the answer in integral form. (d) Use Eq. 6-60. (e) Use Eq. 6-82. (f) Use Eq. 6-79. (g) Use Eq. 6-84.

6-2 (a) Use Eq. 6-60. (b) One way to work this is to notice that $\{\boldsymbol{z} \le z\} = \left\{\dfrac{\boldsymbol{x}}{\boldsymbol{y}} \ge \dfrac{1-z}{z}\right\}$, so if $\boldsymbol{w} = \dfrac{\boldsymbol{x}}{\boldsymbol{y}}$, and $F_w(w)$ is computed from Eq. 6-60, then $F_z(z) = 1 - F_w\left(\dfrac{1-z}{z}\right)$. (c) Consider graphically (as a function of z) where $|x-y| \le z$ inside the square $0 \le x \le a$, $0 \le y \le a$ in the xy-plane. Figure 6-13 is a starting point.

6-3 Begin with a diagram of the region of integration—the graphical approach.

6-4 Why is this a problem at all? Part (a) is Example 6-15; part (b) is example 6-14; and part (c) is obvious from the discussion of joint normality on p.202.

6-5 See Problem 6-1, part (c), or go directly to Eq. 6-148.

6-6 Use Eq. 6-148.

6-7 (a) Use a graphical approach. (b) Use Eq. 6-148. (c) Use Eq. 6-60. (d) Back to a graphical approach.

6-8 Use a graphical approach.

6-9 (a) Eq. 6-60 could be used, but it is easier to use a graphical approach. (b) Use Eq. 6-148.

6-10 The graphical approach is best. Be sure to get the correct triangle in the xy-plane. It has unit area.

6-11 This would be a challenging problem, except that it is (mostly) done in Example 6-27. There, the claim is made about the marginal densities of $\boldsymbol{x}+\boldsymbol{y}$ and $\boldsymbol{x}/\boldsymbol{y}$ that is not quite demonstrated. The best way to attack part (a) is to use the Convolution Theorem on p. 216 along with the characteristic function for a gamma random variable from Table 5-2. (b) may then be attacked with Example 6-27. (c) From Problem 6-2(b) we learned that if $\boldsymbol{u} = \boldsymbol{x}/(\boldsymbol{x}+\boldsymbol{y})$ and $\boldsymbol{w} = \boldsymbol{y}/\boldsymbol{x}$, then $F_u(u) = 1 - F_w\left(\frac{1-u}{u}\right)$.

6-12 Use Eq. 6-119, plus a graphical approach to find where the density is nonzero in the zw-plane.

6-13 Use Eqs. 4-44 and 6-60.

6-14 Use Eq. 6-43.

6-15 Use Eq. 6-43.

6-16 Do not attempt to use the function $g^{-1}(.)$; it may not exist.

6-17 (a) **I**, **M** Use Eq. 6-43 to work the general problem of the sum of independent normal random variables with zero means. (b) Use Eq. 6-62.

6-18 **I**, **M** Start with Eq. 6-148 and reverse the roles of x and y.

6-19 Use Eq. 6-59. The development of the third absolute moment of a zero-mean normal random variable on p. 148 will be useful in evaluating the integral.

6-20 (a) Use Eqs. 5-18 and 6-43. (b) Use Eq. 6-54. (c) Use Eq. 6-60. (d) Use the equation following Eq. 6-78. (e) Use (part of) Example 6-18.

6-21 M First solve the general problem of finding the density of $\boldsymbol{z} = |\boldsymbol{x} - \boldsymbol{y}|$.

6-22 (a) **I**, **M** Compare with Problem 6-17. (b) **I**, **M** Use Eq. 6-43. The integral here is evaluated with Cauchy's residue theorem.

6-23 Use Eqs. 4-39, 5-18, 5-60, and 4-35.

6-24 This is simply Example 6-21 in very thin disguise.

6-25 F See Special Note 2. There is confusion here equivalent to a typographical error. The easiest way to rectify this is to change the problem to find the probability that the combined lifetime exceeds $2/\lambda$ (instead of 2λ) and the probability that the excess lifetime of the second bulb over that of the first exceeds $1/\lambda$ (instead of λ).

6-26 (a) Note that $\boldsymbol{r} = |\boldsymbol{x} - \boldsymbol{y}|$, and Problem 6-2c applies. (b) Note that $\boldsymbol{s} = \boldsymbol{x} + \boldsymbol{y}$.

6-27 (a) Note that $\boldsymbol{z} = \boldsymbol{y}/\boldsymbol{x}$ if $\boldsymbol{y} < \boldsymbol{x}$, or $\boldsymbol{z} = 1$ if $\boldsymbol{y} \geq \boldsymbol{x}$, and be sure to explicitly consider the discontinuity at $z = 1$. (b) Use an approach similar to that for part (a).

6-28 See the hint for Problem 6-11c. Problem 6-1d may also be useful.

6-29 Use Example 6-18 to find $f_z(z)$. Note that $\boldsymbol{w} = |\boldsymbol{x} - \boldsymbol{y}|$, and, starting with Problem 6-2c, find $f_w(w)$. Then use Eq. 6-115 to find $f_{zw}(z, w)$. Finally, test Eq. 6-20.

6-30 (a) Define $\boldsymbol{u} = \boldsymbol{x} + \boldsymbol{y}$. Use Eq. 6-45. Consider two cases: $0 < u < \beta$ and $\beta < u < 2\beta$. In the first case, use the definition of the beta function, Eq. 4-49, to simplify the result. In the second case, simplification is not practical, so leave the result in integral form. (b) Start from Example 6-21. (c) Find the joint density of $\boldsymbol{v}$ and $\boldsymbol{w}$ from Eq. 6-115. Then, from the joint density, find the marginal densities, and show that the joint density is the product of the marginal densities.

6-31 A workhorse problem. (a) Make use of Examples 6-27 and 6-12. Problem 6-11 comes close. (b) Use Eq. 6-115 to find the joint density and show that it is the product of the marginal densities from part (a). (c) Use Theorem 6-1.

6-32 (a) Let $\boldsymbol{z} = \boldsymbol{x}/|\boldsymbol{y}|$. Use a graphical approach to develop an iterated integral expression for $F_z(z)$, as in Example 6-10. Differentiate the double integral expression with respect to z to get a single integral expression for $f_z(z)$, also as in the cited example, then solve the integral. Repeat this process for $\boldsymbol{w} = |\boldsymbol{x}|/|\boldsymbol{y}|$. (b) Use Eq. 6-43 to show that $\boldsymbol{u} \sim N(0, 2)$. From Example 6-14 see that $\boldsymbol{v}$ is exponential with parameter $1/2$. Then use Eq. 6-115 to find the joint density, and see if it is the product of the marginal densities.

6-33 Use Eq. 6-121.

6-34 (a) & (b) **M** Transform first to $\boldsymbol{r} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$ and $\theta = \tan^{-1}(\boldsymbol{y}/\boldsymbol{x})$, using Example 6-22. Then transform from $\boldsymbol{r}$ and θ to $\boldsymbol{u}$ and $\boldsymbol{v}$ to find the joint p.d.f. (c) Express this random variable in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

6-35 Eq. 6-157 defines the F distribution (see Special Note 3). Use Example 5-10. (b) Use Eq. 5-16 to find the p.d.f. of $\boldsymbol{w} = m\boldsymbol{z}/(m\boldsymbol{z} + n)$, and compare the result to Eq. 4-48.

6-36 **F** Use Eq. 6-115 to find the joint density of $\boldsymbol{z}$ and $\boldsymbol{w}$, then use Eq. 6-10 to find the marginal density of $\boldsymbol{z}$, which will reveal that $\boldsymbol{z}$ is not, in fact, exponential. Go on to find the marginal density of $\boldsymbol{w}$, which is exponential.

6-37 Use Eq. 6-115 to find the joint density of $\boldsymbol{z}$ and $\boldsymbol{w}$, then find the marginal densities from the joint density.

6-38 Use Eqs. 5-33 and 6-148.

6-39 First let $\boldsymbol{v} = a\cos\boldsymbol{y}$; thenUse Eqs. 5-33 and 6-43.

6-40 No hint.

6-41 Use Eqs. 6-43 and 6-148.

6-42 (a) Use the result in Problem 6-40. (b) Let $\boldsymbol{w} = \boldsymbol{x} - \boldsymbol{y}$, and develop $P\{\boldsymbol{w} = n\}$. Treat the cases $n < 0$ and $n \geq 0$ separately.

6-43 This problem is somewhat confused. The given information is that two conditional probabilities are equal to $1/(k+1)$. It is only necessary to assume that the two conditional probabilities are equal; it is then possible to show that they must equal $1/(k+1)$.

6-44 Way 1: Proceed directly, using Problem 6-42 and the identity $\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$, or Way 2: Use the moment generating functions of Eq. 5-117.

6-45 (a) Define $\boldsymbol{z} = \min(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{y}$. Find the joint p.m.f. of $\boldsymbol{z}$ and $\boldsymbol{u}$, then find the two marginal p.m.f.s from the joint p.m.f., and show that the joint p.m.f. is the product of the marginal p.m.f.s. (b) Repeat the procedure of part (a) for $\boldsymbol{z}$ and $\boldsymbol{w}$.

6-46 A direct approach is workable here. Use Eqs. 4-57 and 4-56, along with the results of Problem 6-40.

6-47 I Use Example 6-30 to relate μ_{12} and μ_{21} to σ_1, σ_2, and r.

6-48 While it is possible to find $f_z(z)$, where $\mathbf{z} = \mathbf{xy}$ and work the problem in this way, that is unnecessarily difficult. Instead, just focus on the two quadrants of the xy-plane where $\mathbf{z}$ is negative, and use the independence of $\mathbf{x}$ and $\mathbf{y}$ to compute their probabilities.

6-49 Let $\mathbf{w} = \mathbf{x} - \mathbf{y}$, then use Eq. 5-74.

6-50 Use Eq. 6-159.

6-51 I, **M** A fundamental problem. (a) This is the Schwarz inequality for an inner product space. Those familiar with linear algebra know the standard trick to demonstrate it. Show first that $|E\{\mathbf{xy}^*\}| \le \sqrt{E\{|\mathbf{x}|^2\}E\{|\mathbf{y}|^2\}}$. Start with $E\{|a\mathbf{x} - \mathbf{y}|^2\} \ge 0$, and pick the arbitrary constant a artfully. (b) This is the triangle inequality. Note that $E\{\mathbf{xy}^*\} + E\{\mathbf{x}^*\mathbf{y}\} = 2\Re(E\{\mathbf{xy}^*\})$, where $\Re$ denotes the real part, and $\Re(E\{\mathbf{xy}^*\}) \le |E\{\mathbf{xy}^*\}|$, then use part (a).

6-52 F, **M** You cannot show that $\mathbf{y} = a\mathbf{x} + b$ for each and every outcome of the experiment, for it may not be true. You can show that the set of outcomes for which this is true has probability one. Use Eq. 5-89, the Markov inequality.

6-53 This result follows from Problem 6-52.

6-54 M There are two ways to proceed here, one more direct than the other. For the direct method, begin with $f_n(n) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta(n-k)$. For the less direct method, begin by using the results of Problem 5-37 to find $\Phi_x(\omega)$, and then find $\Phi_z(\omega)$ by using Eq. 6-242.

6-55 Note that $\mathbf{y}$ is a function of $\mathbf{x}$, so that $\mathbf{z}$ may be written as a function of $\mathbf{x}$ alone.

6-56 Use Eqs. 5-100, 5-99, and 6-194.

6-57 Characteristic functions provide the easier approach here. With effort and care, the direct approach is also workable.

6-58 All basic stuff.

6-59 Use Eqs. 5-100, 5-119, 6-193, and 6-194.

6-60 Use Eq. 6-159 directly.

6-61 (a) This is Problem 6-4. Another method is available now: Use Eq. 6-55. (b) Use Eq. 6-205. (c) Use Eq. 6-44.

6-62 It helps to know here that $\Gamma(1/2) = \sqrt{\pi}$. Use the string of equations: 4-39, 5-20, 4-25, 6-205, 6-10, and 4-52.

6-63 **I** (a) Use Eqs. 6-179 and 6-166. (b) **D** A continuation or extension of Problem 6-51. See the solution.

6-64 Use Example 6-41.

6-65 **M** Rederive the result in note 4 on p. 337 of the Text. Fix the typo there, where the second equation should read $\mathrm{Var}\{E\{\boldsymbol{x}|\boldsymbol{y}\}\} = E\{(E\{\boldsymbol{x}|\boldsymbol{y}\})^2\} - (E\{E\{\boldsymbol{x}|\boldsymbol{y}\}\})^2$. Use Eq. 6-241 as a part of this derivation.

6-66 Use Eq. 6-180.

6-67 **F** The correct result is: $E\{\boldsymbol{z}\} = \Sigma_n E\{g(x_n, \boldsymbol{y})|x_n\}p_n$. Define $f(y|x_n) = f_n(y)$, and see that $f(x, y) = \Sigma_n p_n f_n(y)\delta(x - x_n)$ to begin the solution.

6-68 (a) Use Eqs. 6-211, 6-26, 5-55, and 4-29. (b) Use Eqs. 6-238 and 6-243.

6-69 **M** Let $I(\mu) = E\{|\boldsymbol{xy}|\}$ and use Price's theorem (Eqs. 6-200 and 6-201) to show that $\frac{\partial I(\mu)}{\partial \mu} = E\{\mathrm{sgn}\boldsymbol{x}\ \mathrm{sgn}\boldsymbol{y}\}$, then use Eq. 6-64 to find this explicitly. Also find $I(0)$ explicitly. Then use $I(\mu) = I(0) + \int_0^{\mu} \frac{\partial I(\mu)}{\partial \mu} d\mu$.

6-70 See Example 6-41.

6-71 Use the unnumbered equation just above Example 4-45.

6-72 Consider $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$, and $\boldsymbol{w} = \boldsymbol{x}$ as a variable transformation.

6-73 To simplify the computation of the Jacobian of the transformation from $\boldsymbol{x}, \boldsymbol{y}$ to $\boldsymbol{z}, \boldsymbol{w}$, note that $\frac{\partial z}{\partial y} = 0$, so that $\frac{\partial w}{\partial x}$ is unimportant.

6-74 Use Bayes' theorem, Eq. 2-44.

6-75 **M** Start with Eq. 6-152, and use Eq. 5-55. Then make some inspired substitutions in the integral to get to the definition of the beta function, Eq. 4-49. Use Eq. 4-51 to relate to the gamma function, then Eq. 4-36 to evaluate the gamma functions. Note that the result is only valid, of course, for $n > 2$.

6-76 Note the typo: the expression $\beta_y(t|\boldsymbol{y} > t)$ should read $\beta_y(t) = f_y(t|\boldsymbol{y} > t)$. Use the unnumbered equation prior to Eq. 6-225.

6-77 Use the Markov inequality, Eq. 5-89.

6-78 Use Eq. 6-159.

6-79 Use Eq. 6-227, Example 6-38, and Eqs. 6-228 and 6-205.

7) Chapter 7

7-1 Rewrite the probability $P\{x_1 < \boldsymbol{x} < x_2,\ y_1 < \boldsymbol{y} < y_2,\ z_1 < \boldsymbol{z} < z_2\}$ in terms of $F(x, y, z)$.

7-2 The useful term "zero-one random variable" was removed from the Fourth Edition, except for this problem, but see Example 4-8. Use Problem 2-15 to show that the zero-one random variables are independent if and only if the events they are associated with are independent.

7-3 **I** Extend the argument in the Text that results in Eqs. 7-57 and 7-58 to the case with nonzero means to find $f(x, y, z)$.

7-4 Expand the polynomial and consider each term.

7-5 (a) Extend the argument leading to Eq. 6-31 to three dimensions. (b) See the bottom of page 259; ultimately use Eq. 5-78.

7-6 **I**, **M** Construct the covariance matrix; it must have a nonnegative determinent.

7-7 **D** Keeping track of the random variables here is hard. Define the three functions: $g(x_3) = E\{\boldsymbol{x}_1\boldsymbol{x}_2|x_3\}$, $h(x_2, x_3) = E\{\boldsymbol{x}_1\boldsymbol{x}_2|x_2, x_3\}$, $p(x_3) = E\{h(\boldsymbol{x}_2, \boldsymbol{x}_3)|x_3\}$. Show that $g(x_3) = p(x_3)$, and interpret.

7-8 **M** Solve for a_1 and a_2 in terms of the expectations of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{y}$. Let $\hat{E}\{\boldsymbol{y}|x_1\} = bx_1$, and solve for b in terms of the expectations of $\boldsymbol{x}_1$, and $\boldsymbol{y}$. Let $\hat{E}\{a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2|x_1\} = cx_1$, and solve for c in terms of a_1 and a_2 and expectations involving $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. Then show $b = c$.

7-9 Use $E\{\boldsymbol{s}^2\} = E\{E\{\boldsymbol{s}^2|\boldsymbol{n}\}\}$. Use Problem 6-51a to limit $(E\{\boldsymbol{x}_i\boldsymbol{x}_j\})^2$. The fact that

$\boldsymbol{x}_i \geq 0$ is not needed.

7-10 M Let event A_n be that heads <u>first</u> appears on the n'th tossing. Note that $[A_1, A_2, \ldots]$ is a partition. Use total probability to find $E\{\boldsymbol{x}_1\}$. Next, establish $E\{\boldsymbol{x}_m | x_{m-1}\}$ by a similar argument, where event B_n is that the first appearance of heads after toss x_{m-1} is on toss $(x_{m-1} + n)$. Finally, use $E\{\boldsymbol{x}_m\} = E\{E\{\boldsymbol{x}_m | \boldsymbol{x}_{m-1}\}\}$.

7-11 Use $\Phi(\omega) = E\{e^{j\omega\boldsymbol{m}}\} = E\{E\{e^{j\omega\boldsymbol{m}} | \boldsymbol{n}\}\}$ to find the characteristic function of $\boldsymbol{m}$ (using the hint in the Text). Compare the result with the characteristic function of a Poisson process, obtained from Eq. 5-119.

7-12 Use the semi-discrete version of Eq. 6-213 (i.e., $f(s) = \Sigma_n f(s|n) p_n$), plus Example 7-1 and Eq. 7-5 (generalized) to find $f(s|n)$.

7-13 No hint.

7-14 Use Example 7-2.

7-15 Note that the event $\{z < \boldsymbol{z} \leq z + dz, w < \boldsymbol{w} \leq w + dw\}$ happens if the events $\{\boldsymbol{x} \leq w\}$ and $\{\boldsymbol{x} > z + dz\}$ do not happen <u>and</u> the events $\{w < \boldsymbol{x} \leq w + dw\}$ and $\{z < \boldsymbol{x} \leq z + dz\}$ happen once, <u>and</u> the event $\{w + dw < \boldsymbol{x} \leq z\}$ happens $(n-2)$ times.

7-16 This follows directly from Eq. 5-96, with independence.

7-17 **M** Show that $(\bar{\boldsymbol{x}}, \boldsymbol{x}_1 - \bar{\boldsymbol{x}}, \ldots, \boldsymbol{x}_n - \bar{\boldsymbol{x}})$ are jointly normal and that $\bar{\boldsymbol{x}}$ is uncorrelated with each $\boldsymbol{x}_i - \bar{\boldsymbol{x}}$. Hence, $\bar{\boldsymbol{x}}$ is independent of the group $(\boldsymbol{x}_1 - \bar{\boldsymbol{x}}, \ldots, \boldsymbol{x}_n - \bar{\boldsymbol{x}})$, and $\bar{\boldsymbol{x}}$ is independent of $\boldsymbol{s}^2$.

7-18 Use Eq. 7-87 to determine α_0, α_1, and α_2. Use Eq. 7-83 to determine the α_1 and α_2 in $\hat{E}\{\boldsymbol{s} - \eta_s | \boldsymbol{x}_1 - \eta_1, \boldsymbol{x}_2 - \eta_2\}$, after you figure out just what this is. See Eq. 7-90.

7-19 Denote $\hat{\boldsymbol{y}} = \hat{E}\{\boldsymbol{y} | \boldsymbol{x}_1, \boldsymbol{x}_2\} = a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2$. Now $\hat{E}\{\hat{E}\{\boldsymbol{y} | \boldsymbol{x}_1, \boldsymbol{x}_2\} | \boldsymbol{x}_1\} = \hat{E}\{\hat{\boldsymbol{y}} | \boldsymbol{x}_1\} = a\boldsymbol{x}_1$. Also, $\hat{E}\{\boldsymbol{y} | \boldsymbol{x}_1\} = b\boldsymbol{x}_1$. The problem here is to show that $a = b$, which follows from the orthogonality principle.

7-20 **M** Use Example 7-2 for $F_x(x)$. Use Problem 7-14 for $F_y(y)$. Use Example 6-21 to obtain $F_{xy}(x, y)$.

7-21 **I**, **D** No conceptual difficulties here, but this is a tedious problem. First deduce that

$\sigma_{\bar{v}}^2$ is unchanged if the means of the $\boldsymbol{x}_i$s are varied, and thus assume that the $\boldsymbol{x}_i$s have zero means. Use brute force to find, in order

$E\{\boldsymbol{x}_i\}$, $E\{\boldsymbol{x}_i\boldsymbol{x}_j\}$, $E\{\boldsymbol{x}_i^2\boldsymbol{x}_j^2\}$, $E\{\bar{\boldsymbol{x}}\boldsymbol{x}_i\}$, $E\{\bar{\boldsymbol{x}}^2\}$, $E\{\boldsymbol{x}_k\boldsymbol{x}_i^2\boldsymbol{x}_j\}$, $E\{\bar{\boldsymbol{x}}\boldsymbol{x}_i^2\boldsymbol{x}_j\}$, $E\{\bar{\boldsymbol{x}}^2\boldsymbol{x}_i\boldsymbol{x}_j\}$, $E\{\bar{\boldsymbol{x}}^3\boldsymbol{x}_i\}$, $E\{\bar{\boldsymbol{x}}^4\}$

Expand $E\{\bar{\boldsymbol{v}}^2\}$ in terms of the above and solve. Note that $\sigma_{\bar{v}}^2 = E\{\bar{\boldsymbol{v}}^2\} - \sigma^4$.

7-22 Use Problem 6-49.

7-23 D Note that for (not necessarily square) matrices A and B, where the product AB is square, that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, where $\mathrm{tr}(\ )$ is the matrix trace function. For a scalar product x, $\mathrm{tr}(x) = x$. Another approach is to note that $R = ADA^t$, where $A^t = A^{-1}$ and D is a diagonal matrix with positive diagonal entries. Define $\boldsymbol{Y} = \boldsymbol{X}AD^{-1/2}A^t$, and show that $E\{\boldsymbol{Y}\boldsymbol{Y}^t\} = E\{\boldsymbol{X}R^{-1}\boldsymbol{X}^t\} = \mathrm{tr}(E\{\boldsymbol{Y}^t\boldsymbol{Y}\})$.

7-24 F This problem cannot be solved as set; more information about the $\boldsymbol{x}_i$s is needed. Assume the sequence $\{\boldsymbol{x}_i\}$ is such that the central limit theorem holds. Among other things, this assures that $\sigma_1^2 + \ldots + \sigma_n^2 \to \infty$ as $n \to \infty$. Use the discussion prior to Eq. 5-35.

7-25 M If you are a mathematician, there is no trouble here. If not, this problem is dificult because you likely have little experience with "ε, N" limit proofs. You must show that, given any $\varepsilon > 0$, there exists an N such that $E\{|\boldsymbol{x}_n - a|^2\} < \varepsilon$ for all $n \geq N$. Since $a_n \to a$, we know that, given any ε_1, there exists an N_1 such that $|a_n - a| < \varepsilon_1$ for all $n \geq N_1$. Since $E\{|\boldsymbol{x}_n - a_n|^2\} \to 0$, we know that, given any ε_2, there exists an N_2 such that $E\{|\boldsymbol{x}_n - a_n|^2\} < \varepsilon_2$ for all $n \geq N_2$. Use the triangle inequality, Problem 6-51b, to relate what is known to what is needed.

7-26 F, M Prove only that if the limit of $E\{\boldsymbol{x}_n\boldsymbol{x}_m\}$ exists, then $\boldsymbol{x}_n$ converges in the mean square. The converse cannot be shown.

7-27 Use the Cauchy criterion, Eq. 7-117. Define $\alpha_n = \sum_{k=1}^{n} \sigma_k^2$. Show that $\{\alpha_n\}$ converges, and use this fact to show that $\{\boldsymbol{y}_n\}$ converges in the mean square sense.

7-28 Let $f_n(y)$ be the density of $\boldsymbol{y}_n = \boldsymbol{y}_{n-1} + \boldsymbol{x}_n$. Use Eq. 6-43 to relate $f_n(\ )$ to $f_{n-1}(\ )$ and to $f_x(\ )$. Solve for the first few densities, compare with the Erlang density (Eq. 4-37) and guess the general result. Then confirm the general result by induction.

7-29 Use Problem 7-28 and the central limit theorem.

7-30 No hint.

7-31 Use Problem 6-22b to show that the sum of Cauchy random variables is Cauchy and never becomes gaussian.

7-32 Note that $\boldsymbol{x}$ and $\boldsymbol{y}$ are presumed to be normal. Use Goodman's theorem, Eq. 7-62, even though $\boldsymbol{x}$ and $\boldsymbol{y}$ are scalars.

8) Chapter 8

Chapter 8 is omitted here.

9) Chapter 9

9-1 (a) From Eqs. 4-74, 5-44, and 5-47, deduce that if $[A_1, \ldots, A_n]$ is a partition, then $E\{\boldsymbol{x}\} = E\{\boldsymbol{x}|A_1\}P(A_1) + \ldots + E\{\boldsymbol{x}|A_n\}P(A_n)$. (b) Since $\boldsymbol{x}(t)$ has only two values (for any fixed t), then $P\{\boldsymbol{x}(t) \le x\}$ can have only three different values, including zero and one.

9-2 Find $F(x, t)$ and differentiate to get $f(x, t)$.

9-3 See Example 9-5. In part (c), notice that an event like $\{\boldsymbol{x}(2) = 2, \boldsymbol{x}(4) = 4\}$ is the same as the event $\{\boldsymbol{x}(4) - \boldsymbol{x}(2) = 2, \boldsymbol{x}(2) = 2\}$, and that the two sub-events that make up this latter form are independent, as they count Poisson points in nonoverlapping intervals. The computation is tedious.

9-4 (b) **M** See: Papoulis, A., *The Fourier Integral and Its Applications*, McGraw Hill, 1962, Appendix I, Eq. I-29.

9-5 Use Eq. 6-63.

9-6 **I**, **M**, **F** You need to show that $\boldsymbol{w}(t) = \int_0^t (t-s)\boldsymbol{v}(s)ds$, for $t \ge 0$. See the solution for how this is done. Note the typo. The correct result is $E\{\boldsymbol{w}^2(t)\} = \int_0^t (t-s)^2 q(s)ds$.

9-7 (a) Use inequality 5-89. (b) Let $\boldsymbol{x}_1 = \boldsymbol{x}(t+\tau)$, $\boldsymbol{x}_2 = \boldsymbol{x}(t)$. Find the region D_x of the

x_1x_2-plane where $|x_1 - x_2| \geq a$, and integrate the second-order density over D_x.

9-8 No hint.

9-9 Do not forget the complex conjugate in Eq. 9-30.

9-10 Use Eq. 7-61.

9-11 **B**, **I**, **M** The Text does not fairly prepare you to solve this problem. In the discussion of linear, constant-coefficient, differential equations, beginning on page 404, the Text notes that such equations are not uniquely solvable without initial conditions. To assure a unique solution and to assure the linearity condition (Eq. 9-86) is satisfied, the Text says that it will presume a solution with "zero" initial conditions at $t = 0$. The trouble with this approach is that the differential equation then only holds for $t \geq 0$, and there is no way that $\mathbf{y}(t)$ can be WSS. By clear implication of this problem, the equation is to hold for all time, and $\mathbf{y}(t)$ will be WSS.

The way to accomplish these goals together is tricky. Suppose we set some arbitrary initial conditions at $t = t_0$, and presume that the differential equation holds for $t \geq t_0$. Now we let $t_0 \to -\infty$. This approach accomplishes all the objectives, provided that the differential equation has a (unique) solution under such assumptions, and the effect of the initial conditions at t_0 on the solution at some fixed t declines to zero as $t_0 \to -\infty$.

This is true for <u>stable</u> differential equations. The differential equation

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = x(t)$$

is stable if and only if the associated polynomial equation in s

$$a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0$$

has only roots with negative real parts (roots in the left half of the s-plane). When this is true, we define the Laplace version of the system function as

$$H(s) = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

The Fourier version of the system function is given by $H(\omega) = H(j\omega)$, and the impulse response function of this linear system is the inverse Fourier transform of the system function, or

$$h(t) \leftrightarrow H(\omega), \quad h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}d\omega$$

Also, in this problem assume $E\{\boldsymbol{v}(t)\} = 0$. To make things more manageable, define the zero-mean process $\boldsymbol{z}(t) = \boldsymbol{y}(t) - 2$, and begin work with it.

9-12 Just note that $f(t_1)g(t_2)\delta(t_1 - t_2) = f(t_1)g(t_1)\delta(t_1 - t_2) = f(t_2)g(t_2)\delta(t_1 - t_2)$.

9-13 **M** If $\varphi(\tau)$ is the phase angle of $R_{xy}(\tau)$, consider $E\{|\boldsymbol{x}(t+\tau) \pm e^{j\varphi(\tau)}\boldsymbol{y}(t)|^2\}$.

9-14 Show that $\boldsymbol{x}(t) = \boldsymbol{y}(t)$ in the mean square sense. Then use the Schwarz inequality, Eq. 9-176, twice to show $R_{xx}(\tau) = R_{xy}(\tau)$ and $R_{yy}(\tau) = R_{xy}(\tau)$.

9-15 No hint.

9-16 Presume φ to be real. Use $\Phi(1) = 0$ to establish $E\{\cos\varphi\} = E\{\sin\varphi\} = 0$, and use $\Phi(2) = 0$ to establish $E\{\cos 2\varphi\} = E\{\sin 2\varphi\} = 0$. Then show $\eta(t) = 0$, and $R(t_1, t_2) = \frac{1}{2}\cos[\omega(t_1 - t_2)]$.

9-17 (a) **B** There is a missing definition here. The stochastic process $\boldsymbol{x}(t)$ has *orthogonal increments* if, for $t_a \le t_b \le t_c \le t_d$, $E\{[\boldsymbol{x}(t_d) - \boldsymbol{x}(t_c)][\boldsymbol{x}(t_b) - \boldsymbol{x}(t_a)]\} = 0$. Using this definition, substitute $t_d = t_2$, $t_c = t_b = t_1$, and $t_a = 0$. (b) **F** You need to show that $\eta_y(t)$ is constant, and you cannot do this unless you assume that $\eta_x(t)$ is constant. Use part (a) to find $R_{yy}(t_1, t_2)$ as a function of $|t_1 - t_2|$.

9-18 No hint.

9-19 See Problem 5-14.

9-20 **I** Note that $f_y(y_1, \ldots, y_n; t_1, \ldots, t_n) = \int_{-\infty}^{\infty} f_{y|\varepsilon}(y_1, \ldots, y_n; t_1, \ldots, t_n|\varepsilon)f_\varepsilon(\varepsilon)d\varepsilon$, and that $f_{y|\varepsilon}(y_1, \ldots, y_n; t_1, \ldots, t_n|\varepsilon) = f_x(y_1, \ldots, y_n; t_1 - \varepsilon, \ldots, t_n - \varepsilon)$.

9-21 Use Eqs. 9-101 and 9-106.

9-22 (b) Note that $f_x(x, t) = f_x(x)$. Note that $\boldsymbol{z}$ and $\boldsymbol{w}$ are jointly normal. Use Eq. 9-46 to find their correlation coefficient.

9-23 **F** You must assume here that $\boldsymbol{x}(t)$ is WSS (or, equivalently, that $\boldsymbol{x}(t)$ has a constant mean). The discussion following Eq. 9-87 assures that $\boldsymbol{x}'(t)$ is normal. Eq. 9-106 will help to find $\sigma^2_{x'}$.

9-24 **M** See the hint in the problem, which is: Use Eq. 9-80, and establish the Fourier series $\sin^{-1} z = \sum_{n=1}^{\infty} \frac{1}{n}[J_0(n\pi) - (-1)^n]\sin n\pi z$.

9-25 Presume, as is implied by the problem, that $\boldsymbol{x}(t)$ is WSS. There are two ways to proceed. One way is direct, but algebraically demanding. The other way is less direct, but easier. The direct method simply relies on the relationships

$$E\{g(\boldsymbol{x}(t))\} = \int_{-\infty}^{\infty} g(x)f(x, t)dx$$

$$E\{g(\boldsymbol{x}(t_1), \boldsymbol{x}(t_2))\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1, x_2)f(x_1, x_2; t_1, t_2)dx_1 dx_2$$

which are obvious extensions of Eqs. 9-7 and 9-8. The first- and second-order density function are obtained from the fact that $\boldsymbol{x}(t)$ is normal. The integration is difficult, but doable. The less direct method relies on the close connection between the desired expected values and the first- and second-order characteristic functions of $\boldsymbol{x}(t)$, which are fully developed for a normal process in Eqs. 5-100 and 6-195. In either case, one must note that $\sigma^2(t) = R_x(0)$, and $r(t+\tau, t) = r(\tau) = R_x(\tau)/R_x(0)$.

9-26 (b) **F** Assume $\tau R_x(\tau) \to 0$ as $\tau \to 0$, (not just $R_x(\tau) \to 0$, as in the problem statement).

9-27 Write $\boldsymbol{y}(t_1)\boldsymbol{y}(t_2)$ as a double integral and take expected values.

9-28 (a) Write $\boldsymbol{y}^2(t)$ as a double integral and take expected values. (b) **I**, **M** Find the general, formal, solution to this differential equation (an equation which is not time-invariant). Then proceed as in part (a).

9-29 (a) Write a formal solution for $\boldsymbol{y}(t)$ to find the impulse response function $h(t)$. Then use Eq. 9-99.

9-30 No hint.

9-31 **F** Assume that $\boldsymbol{x}(t)$ is WSS. Use a method analogous to that used to develop Eq. 9-59.

9-32 (a) Use the solution to Problem 9-29, or otherwise find the impulse response function. Use Eqs. 9-93 and 9-95. (b) Use Example 9-18.

9-33 (a) **M** See Example 5-28.

(b) Use $\cos\omega_0\tau\cos\omega\tau = [\cos(\omega_0 - \omega)\tau + \cos(\omega_0 + \omega)\tau]/2$, with part (a).

9-34 Assume $x(t)$ is real.

9-35 Use the identities $e^{j2a\omega} + e^{-j2a\omega} = 2\cos 2a\omega$, and $1 - \cos 2a\omega = 2\sin^2 2a\omega$.

9-36 No hint.

9-37 **M** Show $\eta_y = I$. Use Eq. 7-61 from Example 7-9 to find R_y, then use Table 9-1 to find S_y.

9-38 Since $S(\omega) \geq 0$, clearly $A = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\left|\Sigma_i a_i e^{j\omega\tau_i}\right|^2 d\omega \geq 0$.

9-39 (a) **M** Use Cauchy's residue theorem to find $R(\tau)$ via contour integration, as in the solution to Problem 5-37. (b) **M** Use the same approach. The pole here is of order two, and the residue is given by $r = \frac{d}{d\omega}[(\omega - \omega_p)^2 h(\omega)]\Big|_{\omega = \omega_p}$, where ω_p is the pole and $h(\omega)$ is the integrand.

9-40 **D** $H(\omega)$ is a complex function of the real variable ω. $H(s)$ is a complex function of the complex variable s. To take the complex conjugate of $H(s)$ you must conjugate both the function $H(\ )$ and the variable s. To assist in the notation, define, in the first and second parts of the problem

$$W(s) = H^*(-s^*) = \int_{-\infty}^{\infty} h^*(t)e^{st}dt, \quad W(z) = H^*(1/z^*) = \Sigma_n h^*[n]z^n$$

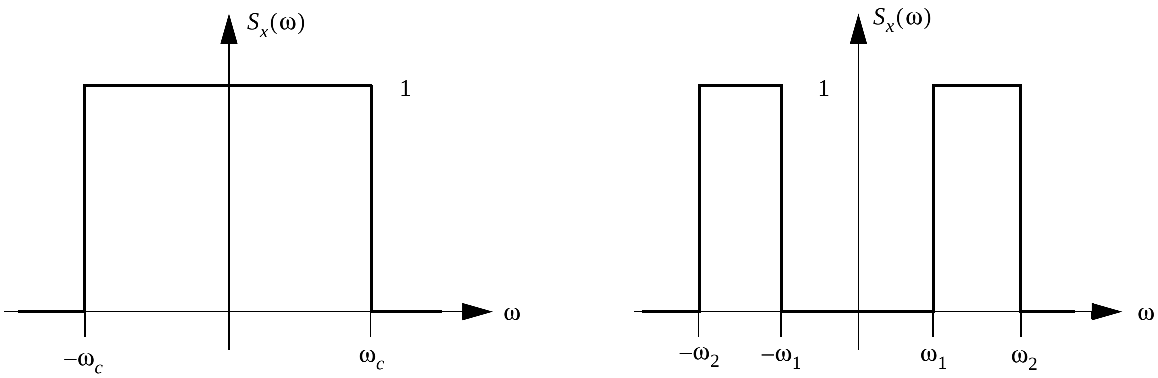
9-41 **B** To solve this problem, it is best to use the general convolution theorem. There are two versions, and both assume three pairs of Fourier transforms: $f(\tau) \leftrightarrow F(\omega)$, $g(\tau) \leftrightarrow G(\omega)$, and $h(\tau) \leftrightarrow H(\omega)$.

I) If $H(\omega) = F(\omega)G(\omega)$, then $h(\tau) = f(\tau) * g(\tau)$

II) If $h(\tau) = f(\tau)g(\tau)$, then $H(\omega) = \frac{1}{2\pi}F(\omega) * G(\omega)$

Notice the factor of two pi in the second version of the theorem. Apparently this factor was neglected in the setting of the problem, leading to the typo. The correct result is $S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi}S_x(\omega) * S_x(\omega)$.

Also, you need to know what ideal LP (low pass) and BP (band pass) spectra are. They are simply unit intensity white noise processes put through an idealized filter. They are illustrated below.



Low Pass Band Pass

9-42 Use Eq. 9-106 to find $R_{xx'}(\tau)$ and $R_{x'x'}(\tau)$. Note that $R_{xx'}(\tau)$ is discontinuous at the origin, leading to a $\delta(\tau)$ term in $R_{x'x'}(\tau)$. Use Table 9-1 to convert from $R_{yy}(\tau)$ to $S_{yy}(\omega)$.

9-43 (a) Use Example 9-27a. (b) Use Eq. 9-115.

9-44 Note that $R(0)$ must be real and nonnegative, so that it must be that $R(\tau_1) = R(0)e^{j\varphi}$, for some phase angle φ. Define $\omega = \varphi/\tau_1$.

9-45 (a) **M** The presumption is that $\mathbf{x}(t)$ is real. Express $E\{\mathbf{x}(t)\hat{\mathbf{x}}(t)\}$ in terms of $S_{xx}(\omega)$ usinf Eq. 9-160. Attempting to use $R_{xx}(\tau)$ will fail.

(b) **D** Express $\tilde{\mathbf{x}}(t)$ (my symbol for double upside-down hats which I lack) in terms of $\mathbf{x}(t)$ and $h(t)$ via an iterated convolution. Since $H^2(\omega) = -1$, it follows that $\rho(t) = -\delta(t)$. An innovative alternate approach involves the Fourier transform of $\mathbf{x}(t)$ itself, but this method lacks generality.

9-46 Find some $G(\omega)$ such that $S_{yy}(\omega) = S_{xx}(\omega)G(\omega)$. Then find that $\omega = \pm\omega_0$ maximizes $G(\omega)$, and put all the available energy of $S_{xx}(\omega)$ at these frequencies.

9-47 Use inequality 9-181 to show that $S_{xy}(\omega) = 0$ for $\omega \neq \omega_0$. Deduce that

$S_{xy}(\omega) = 2\pi B\delta(\omega - \omega_0)$.

9-48 (a) Use convolution integrals directly. Note R_{yx} instead of R_{xy}.

9-49 Presume that over sufficiently small intervals of the frequency axis that the functions $S_{xy}(\omega)$, $S_{xx}(\omega)$, and $S_{yy}(\omega)$ are essentially constant. Use Eq. 9-181.

9-50 Presume that $\mathbf{x}(t)$ is real. Use the cosine inequality, Eq. 6-169.

9-51 Proceed as in Problem 9-50. Note the typo: the greater-than sign should be a greater-than-or-equal sign.

9-52 Express $R[m]$ in terms of $f(\omega)$. Compare with Eq. 9-193.

9-53 (a) **I**, **M** Find the solutions to the homogeneous differential equation. Use the variation of parameters method to find the solution to the non-homogeneous differential equation. Write $\mathbf{y}^2(t)$ as a double integral and take expected values.

(b) **I**, **D** Use the same approach as above for the difference equation.

9-54 (a) **I** Find the general solution to the difference equation.

(b) **I** Modify the development of part (a).

10) Chapter 10

10-1 (a) Use Eq. 9-13. (b) Use Eq. 10-52 and Example 5-28.

10-2 Use Eq. 10-52 for $f_x(x, t)$ and $f_y(y, t)$. Then use Eq. 6-70 for $f_z(z, t)$.

10-3 **I** Find the differential equation relating $\mathbf{v}(t)$ and $\mathbf{n}_e(t)$. Use Laplace transforms to find $H(s)$, and then solve for $|H(\omega)|^2$. Use Eq. 10-74 to find $S_v(\omega)$. A similar process works for the current case.

10-4 Note that $e^{-2\alpha t}U(t) \leftrightarrow \dfrac{1}{(j\omega + 2\alpha)}$, and, if $h(t) \leftrightarrow H(\omega)$, then $h'(t) \leftrightarrow j\omega H(\omega)$.

10-5 Use Example 7-12 and Eq. 9-106.

10-6 Presume that the Wiener process, $\mathbf{w}(t)$, is a normal stochastic process with zero mean. Use Eq. 6-199 to find R_y and Problem 6-69 to find R_z.

10-7 Use Campbell's theorem, Eq. 10-102. Note that $\mathbf{s}(7)$ can be zero only if no Poisson points arrive in the interval prior to $t_0 = 7$ when $h(7-t)$ is not zero.

10-8 Show that $S_{xy}(\omega)$ is purely imaginary. Then use Eqs. 9-147 and 9-149 to confine $H(\omega)$.

10-9 D A brute force approach works here, but it is both lengthy and demanding.

10-10 F As the Text points out on the bottom of page 465, given just $\mathbf{x}(t)$ there is no unambiguous complex envelope. You must assume Rice's representation, so that the $\mathbf{y}(t)$ in Eq. 10-132 is $\hat{\mathbf{x}}(t)$.

10-11 Differentiate the relationship of Eq. 9-134 to find $R_{xx}''(0)$. Use Eqs. 10-148 and 10-150.

10-12 F You must assume that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real, jointly WSS processes, and that you are given the constant ω.

10-13 M First note the typo. The correct answer is

$$S_{xx}(\omega) = \frac{2\pi}{T^2}\left|\int_0^T f(t)e^{-j\omega t}dt\right|^2 \sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi m}{T}\right)$$

Begin by showing that the stochastic process $\mathbf{y}(t) = f(t)$ is SSCS. Then show $\mathbf{x}(t)$ is SSS, and use Eq. 10-177 for $R_{xx}(\tau)$. Find $S_{xx}(\omega)$ directly from Eq. 9-133, then split up the infinite time integral into a sum over consecutive spans of length T. Finally, use Eq. 10A-2 to get the desired result.

10-14 M Mimic Eq. 10-197 and define

$$\mathbf{y}_N(t) = \mathbf{x}(t+\tau) - \sum_{n=-N}^{N} \mathbf{x}(t+nT)\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}, \quad T = \pi/\sigma$$

Note that $\varepsilon_N(t) = \mathbf{y}_N(0)$.

10-15 Assume $\mathbf{x}(t)$ is real, and use the proof at the bottom of page 477.

10-16 Use Eq. 10-196, Eq. 10-198 with $\omega = 0$, and Table 9-1.

10-17 See hint 10-16 above.

10-18 Note that in the range $|\tau| < \frac{\pi}{2\sigma}$ and $\omega \le \sigma$, then $\cos\omega\tau \ge \cos\sigma\tau$. Also, assume that $\boldsymbol{x}(t)$ is real, as usual in BL problems.

10-19 M This is a straightforward application of the "Papoulis Sampling Expansion" on Text page 480, but difficult because it involves considerable manipulation. For some help, note that $P_1(\omega, \tau) = 1 - \frac{\omega}{\sigma}(e^{j\sigma\tau} - 1)$, $P_2(\omega, \tau) = \frac{1}{j\sigma}(e^{j\sigma\tau} - 1)$, and $p_1(\tau) = \frac{1}{\sigma}\int_{-\sigma}^{0}\left[1 - \frac{\omega}{\sigma}(e^{j\sigma\tau} - 1)\right]e^{j\omega\tau}d\omega$, $p_2(\tau) = \frac{1}{j\sigma^2}(e^{j\sigma\tau} - 1)\int_{-\sigma}^{0} e^{j\omega\tau}d\omega$. Use Eq. 10-206 with $t_0 = 0$, noting that $\boldsymbol{y}_1(n\bar{T}) = \boldsymbol{y}_1(n\Delta) = \boldsymbol{x}(n\Delta)$, $\boldsymbol{y}_2(n\bar{T}) = \boldsymbol{y}_2(n\Delta) = \boldsymbol{x}'(n\Delta)$.

10-20 Use the method of Eq. 10-212, with $f(t) = \cos\omega_0 t \cos\omega t$, and with the integral limits from $-a$ to a. Get equivalent formulas to those following Eq. 10-214.

10-21 F First correct the typo. The sum should be $\boldsymbol{X}_c(\omega) = \frac{1}{\lambda}\sum_{\substack{i \\ |t_i| < a}} \boldsymbol{x}(\boldsymbol{t}_i)e^{-j\omega\boldsymbol{t}_i}$. Next, you must assume that $\boldsymbol{x}(t)$ is independent of the process $\boldsymbol{z}(t) = \sum_i \delta(t - \boldsymbol{t}_i)$. The integral transform of Eq. 9-59, valid for any function $C(\tau)$, is also useful.

10-22 No hint.

10-23 Consider $I = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_i b_j^* - a_j b_i^*\right|^2 \ge 0$.

10-24 (a) **D** The treatment of discrete processes in the Text is too brief to even give a useful hint here, except to mimic the matched filter discussion for a continuous process. Assume real processes.

(b) **D** Note that $\boldsymbol{v}[n]$ is not presumed to be white here. To maximize $r = \frac{y_f^2[0]}{E\{\boldsymbol{y}_v^2[n]\}}$, you should minimize the denominator while holding the numerator constant. Use the method of Lagrange multipliers.

10-25 M Find the impulse response function $h(t)$ (see Example 10-2). From the deterministic input, $f(t) = A\cos\omega_0 t$, find the deterministic output in the form $y_f(t) = B\cos(\omega_0 t + \varphi)$ to relate B to A, α, and ω_0. Find the spectrum of the noise output, and from that find the autocorrelation of the noise output, using Table 9-1. Use Eq. 9-152 to find the average power of the noise output.

10-26 M (a) It helps to define the vectors and matrices: $\vec{a}^T = (a_0, a_1, \ldots, a_m)$,

$$\vec{f}^T = (f_0, f_1, \ldots, f_m) = (f(t_0), f(t_0 - T), \ldots, f(t_0 - mT)) \text{, and } R = \begin{bmatrix} R_{00} & R_{01} & \ldots & R_{0m} \\ R_{10} & R_{11} & \ldots & R_{1m} \\ \ldots & \ldots & & \ldots \\ R_{m0} & R_{m1} & \ldots & R_{mm} \end{bmatrix},$$

where $R_{ij} = R_v(iT - jT) = E\{\boldsymbol{v}(t_0 - iT)\boldsymbol{v}(t_0 - jT)\}$. Then, show

$y_f = y_f(t_0) = \sum_{i=0}^{m} a_i f_i = \vec{a}^T\vec{f}$, and $E = E\{\boldsymbol{y}_v^2(t_0)\} = \vec{a}^T R\vec{a}$. Maximize r by minimizing E subject to the constraint of a given value of $y_f > 0$ by the method of Lagrange multipliers.

(b) The maximum value of r is $\sqrt{\vec{f}^T R^{-1}\vec{f}} = \sqrt{y_f/k}$.

10-27 This follows directly from Eq. 10-243 using (repeatedly) the identity $\int_{-\infty}^{\infty} e^{-j\omega\tau} d\tau = 2\pi\delta(\omega)$.

10-28 M Several facts are useful here. If $t_1 < t_2 < t_3$, then $\tilde{\boldsymbol{x}}(t_3) = \tilde{\boldsymbol{x}}(t_1) + [\tilde{\boldsymbol{x}}(t_2) - \tilde{\boldsymbol{x}}(t_1)] + [\tilde{\boldsymbol{x}}(t_3) - \tilde{\boldsymbol{x}}(t_2)]$, and the three random variables $\tilde{\boldsymbol{x}}(t_1)$, $\tilde{\boldsymbol{x}}(t_2) - \tilde{\boldsymbol{x}}(t_1)$, and $\tilde{\boldsymbol{x}}(t_3) - \tilde{\boldsymbol{x}}(t_2)$ are mean zero and independent, because they count Poisson points in nonoverlapping intervals. From the discussion of Poisson random variables on page 149 of the Text, it follows that $E\{\tilde{\boldsymbol{x}}^3(t_1)\} = \lambda t_1$. Finally, ignore the hint in the problem and use instead the three identities

$$\min(t_1, t_2, t_3) = \min(t_1, \min(t_2, t_3)), \qquad \frac{\partial \min(t_a, t_b)}{\partial t_a} = U(t_b - t_a)$$

$$U(\min(t_2, t_3) - t_1) = U(t_2 - t_1)U(t_3 - t_1)$$

10-29 Follow the outline in the problem.

11) Chapter 11

11-1 D Finding the whitening filter is easy, especially if you note that

$\cos 2\omega = (z^2 + z^{-2})/2$. Finding the autocorrelation sequence $R_{xx}[m]$ is hard—so hard that no hint is given here, see the solution.

11-2 Factor $S(s) = \dfrac{N(s)N(-s)}{D(s)D(-s)}$, then set $L(s) = \dfrac{N(s)}{D(s)}$.

11-3 Use the convolution sum: $\boldsymbol{s}[n] = \sum_{k=0}^{\infty} l_s[k]\boldsymbol{i}[n-k]$.

11-4 (a) **I** Note that $R_{yx}'(\tau) = E\{\boldsymbol{y}'(t+\tau)\boldsymbol{x}^*(t)\}$, etc.

(b) **I**, **M** Use the discussion following Eq. 11-25. Note that $S_{yx}^{+}(s) = S_{yx}(s) = q/D(s)$ to find $R_{yx}(\tau)$ for $\tau > 0$. Note that $S_{yy}^{-}(s) = S_{yy}^{+}(-s)$, and $S_{yy}(s) = \dfrac{q}{D(s)D(-s)}$ to find $S_{yy}^{+}(s)$ and, hence, $R_{yy}(\tau)$ for $\tau > 0$.

11-5 Show $R_{xx}[m] = R_{ss}[m] + R_{vv}[m]$ and $R_{vv}[m] = q\delta[m]$. Deduce that $S_{ss}(z) = \dfrac{1}{D(z)D(1/z)}$, where $D(z)$ has all its roots z_i with $|z_i| < 1$. Conclude that $S_{xx}(z)$ is also rational, with the same poles as $S_{ss}(z)$, and that $S_{xx}(z) = S_{xx}(1/z)$.

11-6 Define $\boldsymbol{s}(t) = \dfrac{1}{n}\sum_{k=1}^{n} \boldsymbol{x}(t+kT)$, and regard $\boldsymbol{s}(t)$ as the output of a linear system with input $\boldsymbol{x}(t)$. Find the impulse response function, the system function, and use Eq. 9-152.

11-7 **D**, **F** Presume $\boldsymbol{x}(t)$ is WSS, so the origin of the time scale may be shifted so that the interval $(0, T)$ in the old time scale corresponds to the interval $(-a, a)$ in the new time scale. Particularize the integral equation 11-58. Differentiate the integral equation twice to obtain a differential equation. Find the general solutions to the differential equation and put them back into the integral equation to learn more about them. Use the normalization equation, Eq. 11-56, to scale the $\varphi(t)$ functions. Note that the results should read: $\beta_n = \left(a + \dfrac{\lambda_n}{2}\right)^{-1/2}$ and $\beta_n' = \left(a + \dfrac{\lambda_n'}{2}\right)^{-1/2}$, not the results given in the Text.

11-8 Write $E\{|\boldsymbol{X}(\omega)|^2\}$ as a double integral. Transform to a single integral as in Eq. 9-59. Differentiate with respect to T.

11-9 No hint.

11-10 No hint.

11-11 M Use Eq. 11-51. (a) Show

$$E\{\boldsymbol{x}(t)\boldsymbol{x}^*(t)\} = E\{\boldsymbol{x}(t)\hat{\boldsymbol{x}}^*(t)\} = E\{\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)\} = E\{\hat{\boldsymbol{x}}(t)\hat{\boldsymbol{x}}^*(t)\} = R(0)$$

In order to do part (a), it is useful to do part (b). (c) Substitute $s = t - \alpha$ in the $\beta_n(\alpha)$ expression.

11-12 As is usual in these cases, show that $E\{\boldsymbol{x}(t)\} = 0$ and that $E\{\boldsymbol{x}(t)\boldsymbol{x}^*(s)\}$ is a function of $t - s$.

11-13 Deduce that if $\boldsymbol{A}$ and $\boldsymbol{B}$ satisfy Eqs. 11-79, then it must be that $E\{\boldsymbol{A}(u)\boldsymbol{A}(v)\} = E\{\boldsymbol{B}(u)\boldsymbol{B}(v)\} = Q(u)\delta(u-v)$ and $E\{\boldsymbol{A}(u)\boldsymbol{B}(v)\} = 0$.

11-14 M Note that $\int_{-T}^{T} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} f(t)p_T(t)e^{-j\omega t}dt$, where

$$p_T(t) = U(t-T) - U(T-t) = \begin{cases} 1 & -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

Use the frequency convolution theorem: if $F_1(\omega) = \int_{-\infty}^{\infty} f_1(t)e^{-j\omega t}dt$, and

$F_2(\omega) = \int_{-\infty}^{\infty} f_2(t)e^{-j\omega t}dt$, then

$$F_{12}(\omega) = \int_{-\infty}^{\infty} f_1(t)f_2(t)e^{-j\omega t}dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(y)F_2(\omega - y)dy$$

Use Eq. 11-82. Note that $\int_0^{\infty} \frac{\sin^2 \alpha T}{\alpha^2} d\alpha = \frac{T\pi}{2}$.

12) Chapter 12

12-1 No hint.

12-2 I, D Note that $\boldsymbol{x}(t)$ is WSS. Use Eq. 12-35 to deduce that $f(x, x; \tau) \to f^2(x)$ as $|\tau| \to \infty$. Find both sides of this limit explicitly, and show that the limit requires

$r(\tau) \to 0$. Use Eq. 12-10.

12-3 Use Eq. 12-27.

12-4 Show that $C_{zz}(\tau)$ does not depend on τ, so that Eq. 12-5 cannot be satisfied.

12-5 **I** Recall that $\lim_{T \to \infty} \boldsymbol{R}_T = R_{xy}(\lambda)$ if and only if both $E\{\boldsymbol{R}_T\} = R_{xy}(\lambda)$, and $\sigma^2_{R_T} \to 0$.

12-6 **F**, **D**, **I** The condition in this problem should read: $R(t+\tau, t) \to \eta(t+\tau)\eta(t)$ as $|\tau| \to \infty$, uniformly in t. Also, you must assume $C(t, t) < \sigma^2$ for some σ and all t. Ignore the hint in the Text. Define the average mean as $\bar{\eta} = \frac{1}{T}\int_0^T \eta(t)dt$. Show that $\lim_{c \to \infty} \frac{1}{2c}\int_{-c}^{c} \eta(t)dt = \bar{\eta}$. Use this result to show that $\lim_{c \to \infty} E\{(\eta_c - \bar{\eta})^2\} = 0$ is equivalent to the revised condition above. Do Problem 13-7 first to understand this latter result.

12-7 **I**, **D**! Assume that $C(t, t) < \sigma^2$ for some σ and all t. Use Eq. 9-37 to get $\sigma_T^2 = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1, t_2)dt_1dt_2$. Change variables to $t = t_2$, $\tau = t_1 - t_2$. Break up the integral in the $t\tau$-plane into portions where $|\tau| \geq T_2$ and $|\tau| < T_2$, and find ways to limit both parts.