

5. Functions of a Random Variable

Let X be a r.v defined on the model (Ω, F, P) , and suppose $g(x)$ is a function of the variable x . Define

$$Y = g(X). \quad (5-1)$$

Is Y necessarily a r.v? If so what is its PDF $F_Y(y)$, pdf $f_Y(y)$?

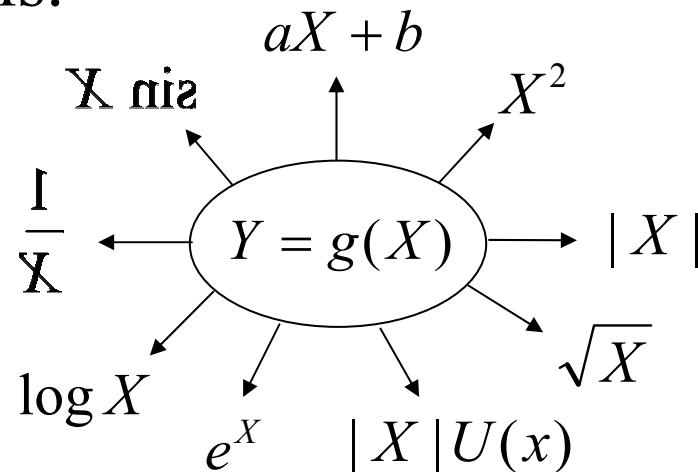
Clearly if Y is a r.v, then for every Borel set B , the set of ξ for which $Y(\xi) \in B$ must belong to F . Given that X is a r.v, this is assured if $g^{-1}(B)$ is also a Borel set, i.e., if $g(x)$ is a Borel function. In that case if X is a r.v, so is Y , and for every Borel set B

$$P(Y \in B) = P(X \in g^{-1}(B)). \quad (5-2)$$

In particular

$$F_Y(y) = P(Y(\xi) \leq y) = P(g(X(\xi)) \leq y) = P(X(\xi) \leq g^{-1}(-\infty, y]). \quad (5-3)$$

Thus the distribution function as well of the density function of Y can be determined in terms of that of X . To obtain the distribution function of Y , we must determine the Borel set on the x -axis such that $X(\xi) \leq g^{-1}(y)$ for every given y , and the probability of that set. At this point, we shall consider some of the following functions to illustrate the technical details.



$$\text{Example 5.1: } Y = aX + b \quad (5-4)$$

Solution: Suppose $a > 0$.

$$F_Y(y) = P(Y(\xi) \leq y) = P(aX(\xi) + b \leq y) = P\left(X(\xi) \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right). \quad (5-5)$$

and

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right). \quad (5-6)$$

On the other hand if $a < 0$, then

$$\begin{aligned} F_Y(y) &= P(Y(\xi) \leq y) = P(aX(\xi) + b \leq y) = P\left(X(\xi) > \frac{y-b}{a}\right) \\ &= 1 - F_X\left(\frac{y-b}{a}\right), \end{aligned} \quad (5-7)$$

and hence

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right). \quad (5-8)$$

From (5-6) and (5-8), we obtain (for all a)

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right). \quad (5-9)$$

Example 5.2: $Y = X^2$. (5-10)

$$F_Y(y) = P(Y(\xi) \leq y) = P(X^2(\xi) \leq y). \quad (5-11)$$

If $y < 0$, then the event $\{X^2(\xi) \leq y\} = \phi$, and hence

$$F_Y(y) = 0, \quad y < 0. \quad (5-12)$$

For $y > 0$, from Fig. 5.1, the event $\{Y(\xi) \leq y\} = \{X^2(\xi) \leq y\}$ is equivalent to $\{x_1 < X(\xi) \leq x_2\}$.

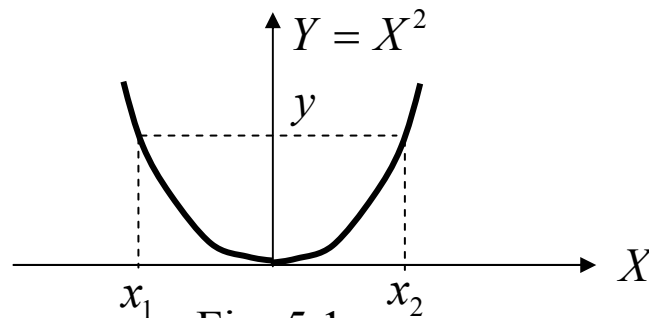


Fig. 5.1

Hence

$$\begin{aligned} F_Y(y) &= P(x_1 < X(\xi) \leq x_2) = F_X(x_2) - F_X(x_1) \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y > 0. \end{aligned} \tag{5-13}$$

By direct differentiation, we get

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})), & y > 0, \\ 0, & \text{otherwise} . \end{cases} \tag{5-14}$$

If $f_X(x)$ represents an even function, then (5-14) reduces to

$$f_Y(y) = \frac{1}{\sqrt{y}} f_X(\sqrt{y}) U(y). \tag{5-15}$$

In particular if $X \sim N(0,1)$, so that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \tag{5-16}$$

and substituting this into (5-14) or (5-15), we obtain the p.d.f of $Y = X^2$ to be

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} U(y). \quad (5-17)$$

On comparing this with (3-36), we notice that (5-17) represents a Chi-square r.v with $n = 1$, since $\Gamma(1/2) = \sqrt{\pi}$. Thus, if X is a Gaussian r.v with $\mu = 0$, then $Y = X^2$ represents a Chi-square r.v with one degree of freedom ($n = 1$).

Example 5.3: Let

$$Y = g(X) = \begin{cases} X - c, & X > c, \\ 0, & -c < X \leq c, \\ X + c, & X \leq -c. \end{cases}$$

In this case

$$P(Y = 0) = P(-c < X(\xi) \leq c) = F_X(c) - F_X(-c). \quad (5-18)$$

For $y > 0$, we have $x > c$, and $Y(\xi) = X(\xi) - c$ so that

$$\begin{aligned} F_Y(y) &= P(Y(\xi) \leq y) = P(X(\xi) - c \leq y) \\ &= P(X(\xi) \leq y + c) = F_X(y + c), \quad y > 0. \end{aligned} \quad (5-19)$$

Similarly $y < 0$, if $x < -c$, and $Y(\xi) = X(\xi) + c$ so that

$$\begin{aligned} F_Y(y) &= P(Y(\xi) \leq y) = P(X(\xi) + c \leq y) \\ &= P(X(\xi) \leq y - c) = F_X(y - c), \quad y < 0. \end{aligned} \quad (5-20)$$

Thus

$$f_Y(y) = \begin{cases} f_X(y + c), & y > 0, \\ [F_X(c) - F_X(-c)]\delta(y), & \\ f_X(y - c), & y < 0. \end{cases} \quad (5-21)$$

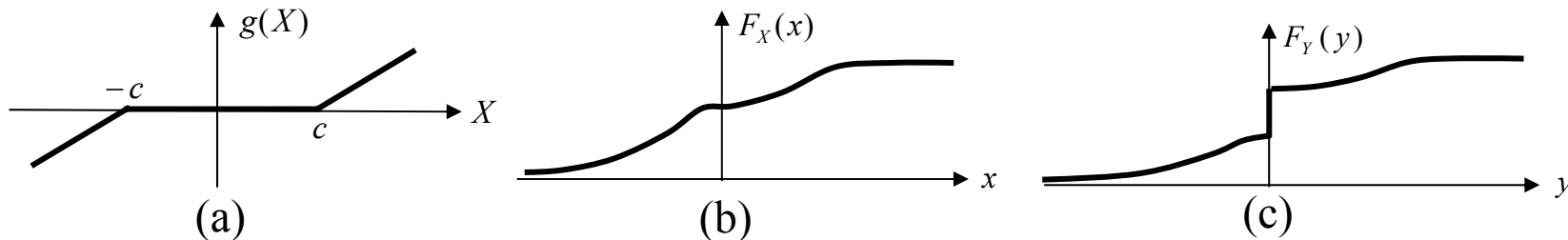


Fig. 5.2

Example 5.4: Half-wave rectifier

$$Y = g(X); \quad g(x) = \begin{cases} x, & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (5-22)$$

In this case

$$P(Y = 0) = P(X(\xi) \leq 0) = F_X(0). \quad (5-23)$$

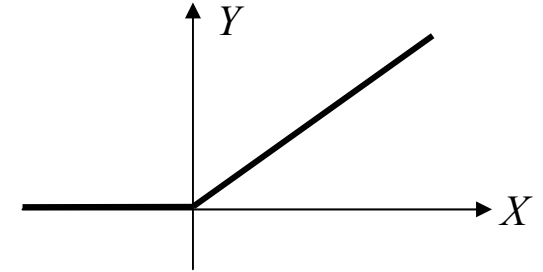


Fig. 5.3

and for $y > 0$, since $Y = X$,

$$F_Y(y) = P(Y(\xi) \leq y) = P(X(\xi) \leq y) = F_X(y). \quad (5-24)$$

Thus

$$f_Y(y) = \begin{cases} f_X(y), & y > 0, \\ F_X(0)\delta(y) & y = 0, \\ 0, & y < 0, \end{cases} = f_X(y)U(y) + F_X(0)\delta(y). \quad (5-25)$$

Note: As a general approach, given $Y = g(X)$, first sketch the graph $y = g(x)$, and determine the range space of y .

Suppose $a < y < b$ is the range space of $y = g(x)$.

Then clearly for $y < a$, $F_Y(y) = 0$, and for $y > b$, $F_Y(y) = 1$, so that $F_Y(y)$ can be nonzero only in $a < y < b$. Next, determine whether there are discontinuities in the range space of y . If so evaluate $P(Y(\xi) = y_i)$ at these discontinuities. In the continuous region of y , use the basic approach

$$F_Y(y) = P(g(X(\xi)) \leq y)$$

and determine appropriate events in terms of the r.v X for every y . Finally, we must have $F_Y(y)$ for $-\infty < y < +\infty$, and obtain

$$f_Y(y) = \frac{dF_Y(y)}{dy} \quad \text{in } a < y < b.$$

However, if $Y = g(X)$ is a continuous function, it is easy to establish a direct procedure to obtain $f_Y(y)$. A continuous function $g(x)$ with $g'(x)$ nonzero at all but a finite number of points, has only a finite number of maxima and minima, and it eventually becomes monotonic as $|x| \rightarrow \infty$. Consider a specific y on the y -axis, and a positive increment Δy as shown in Fig. 5.4

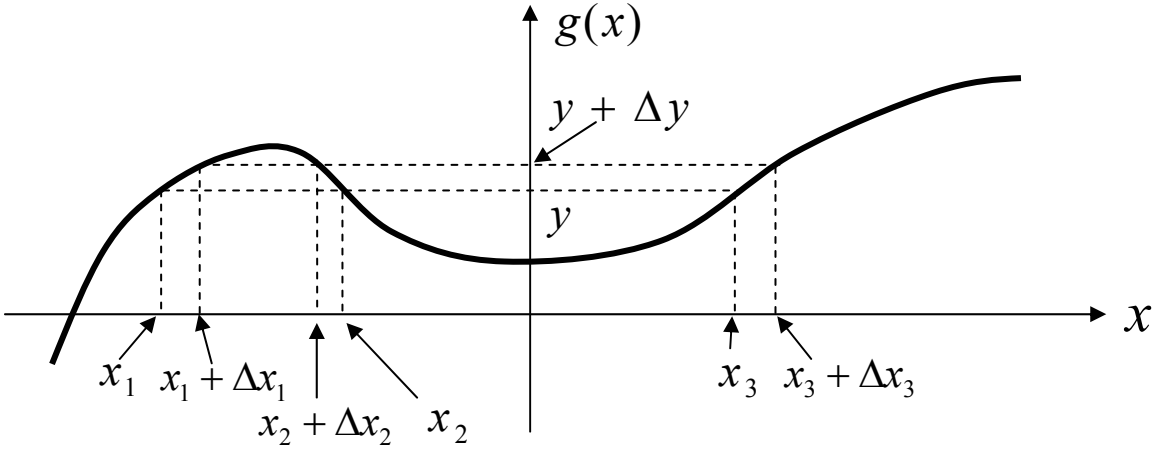


Fig. 5.4

$f_Y(y)$ for $Y = g(X)$, where $g(\cdot)$ is of continuous type.

Using (3-28) we can write

$$P\{y < Y(\xi) \leq y + \Delta y\} = \int_y^{y+\Delta y} f_Y(u) du \approx f_Y(y) \cdot \Delta y. \quad (5-26)$$

But the event $\{y < Y(\xi) \leq y + \Delta y\}$ can be expressed in terms of $X(\xi)$ as well. To see this, referring back to Fig. 5.4, we notice that the equation $y = g(x)$ has three solutions x_1, x_2, x_3 (for the specific y chosen there). As a result when $\{y < Y(\xi) \leq y + \Delta y\}$, the r.v X could be in any one of the three mutually exclusive intervals

$$\{x_1 < X(\xi) \leq x_1 + \Delta x_1\}, \quad \{x_2 + \Delta x_2 < X(\xi) \leq x_2\} \quad \text{or} \quad \{x_3 < X(\xi) \leq x_3 + \Delta x_3\}.$$

Hence the probability of the event in (5-26) is the sum of the probability of the above three events, i.e.,

$$P\{y < Y(\xi) \leq y + \Delta y\} = P\{x_1 < X(\xi) \leq x_1 + \Delta x_1\} \\ + P\{x_2 + \Delta x_2 < X(\xi) \leq x_2\} + P\{x_3 < X(\xi) \leq x_3 + \Delta x_3\}. \quad (5-27)_{11}$$

For small $\Delta y, \Delta x_i$, making use of the approximation in (5-26), we get

$$f_Y(y)\Delta y = f_X(x_1)\Delta x_1 + f_X(x_2)(-\Delta x_2) + f_X(x_3)\Delta x_3. \quad (5-28)$$

In this case, $\Delta x_1 > 0$, $\Delta x_2 < 0$ and $\Delta x_3 > 0$, so that (5-28) can be rewritten as

$$f_Y(y) = \sum_i f_X(x_i) \frac{|\Delta x_i|}{\Delta y} = \sum_i \frac{1}{|\Delta y / \Delta x_i|} f_X(x_i) \quad (5-29)$$

and as $\Delta y \rightarrow 0$, (5-29) can be expressed as

$$f_Y(y) = \sum_i \frac{1}{|dy/dx|_{x_i}} f_X(x_i) = \sum_i \frac{1}{|g'(x_i)|} f_X(x_i). \quad (5-30)$$

The summation index i in (5-30) depends on y , and for every y the equation $y = g(x_i)$ must be solved to obtain the total number of solutions at every y , and the actual solutions x_1, x_2, \dots all in terms of y .

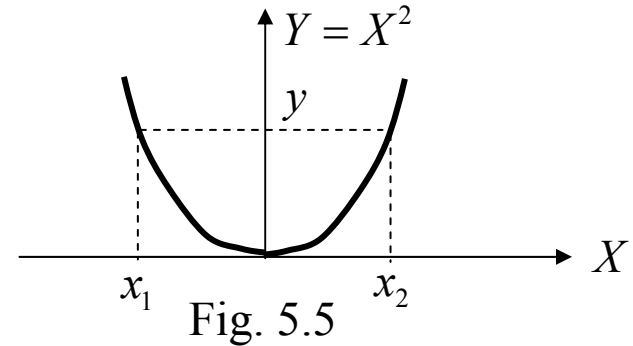
For example, if $Y = X^2$, then for all $y > 0$, $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$ represent the two solutions for each y . Notice that the solutions x_i are all in terms of y so that the right side of (5-30) is only a function of y . Referring back to the example $Y = X^2$ (Example 5.2) here for each $y > 0$, there are two solutions given by $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$. ($f_Y(y) = 0$ for $y < 0$).

Moreover

$$\frac{dy}{dx} = 2x \quad \text{so that} \quad \left| \frac{dy}{dx} \right|_{x=x_i} = 2\sqrt{y}$$

and using (5-30) we get

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})), & y > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (5-31)$$



which agrees with (5-14).

Example 5.5: $Y = \frac{1}{X}$. Find $f_Y(y)$. (5-32)

Solution: Here for every y , $x_1 = 1/y$ is the only solution, and

$$\frac{dy}{dx} = -\frac{1}{x^2} \quad \text{so that} \quad \left| \frac{dy}{dx} \right|_{x=x_1} = \frac{1}{1/y^2} = y^2,$$

and substituting this into (5-30), we obtain

$$f_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right). \quad (5-33)$$

In particular, suppose X is a Cauchy r.v as in (3-39) with parameter α so that

$$f_X(x) = \frac{\alpha / \pi}{\alpha^2 + x^2}, \quad -\infty < x < +\infty. \quad (5-34)$$

In that case from (5-33), $Y = 1/X$ has the p.d.f

$$f_Y(y) = \frac{1}{y^2} \frac{\alpha / \pi}{\alpha^2 + (1/y)^2} = \frac{(1/\alpha) / \pi}{(1/\alpha)^2 + y^2}, \quad -\infty < y < +\infty. \quad (5-35)$$

But (5-35) represents the p.d.f of a Cauchy r.v with parameter $1/\alpha$. Thus if $X \sim C(\alpha)$, then $1/X \sim C(1/\alpha)$.

Example 5.6: Suppose $f_X(x) = 2x/\pi^2$, $0 < x < \pi$, and $Y = \sin X$. Determine $f_Y(y)$.

Solution: Since X has zero probability of falling outside the interval $(0, \pi)$, $y = \sin x$ has zero probability of falling outside the interval $(0, 1)$. Clearly $f_Y(y) = 0$ outside this interval. For any $0 < y < 1$, from Fig.5.6(b), the equation $y = \sin x$ has an infinite number of solutions $\dots, x_1, x_2, x_3, \dots$, where $x_1 = \sin^{-1} y$ is the principal solution. Moreover, using the symmetry we also get $x_2 = \pi - x_1$ etc. Further,

$$\frac{dy}{dx} = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$$

so that

$$\left| \frac{dy}{dx} \right|_{x=x_i} = \sqrt{1 - y^2}.$$

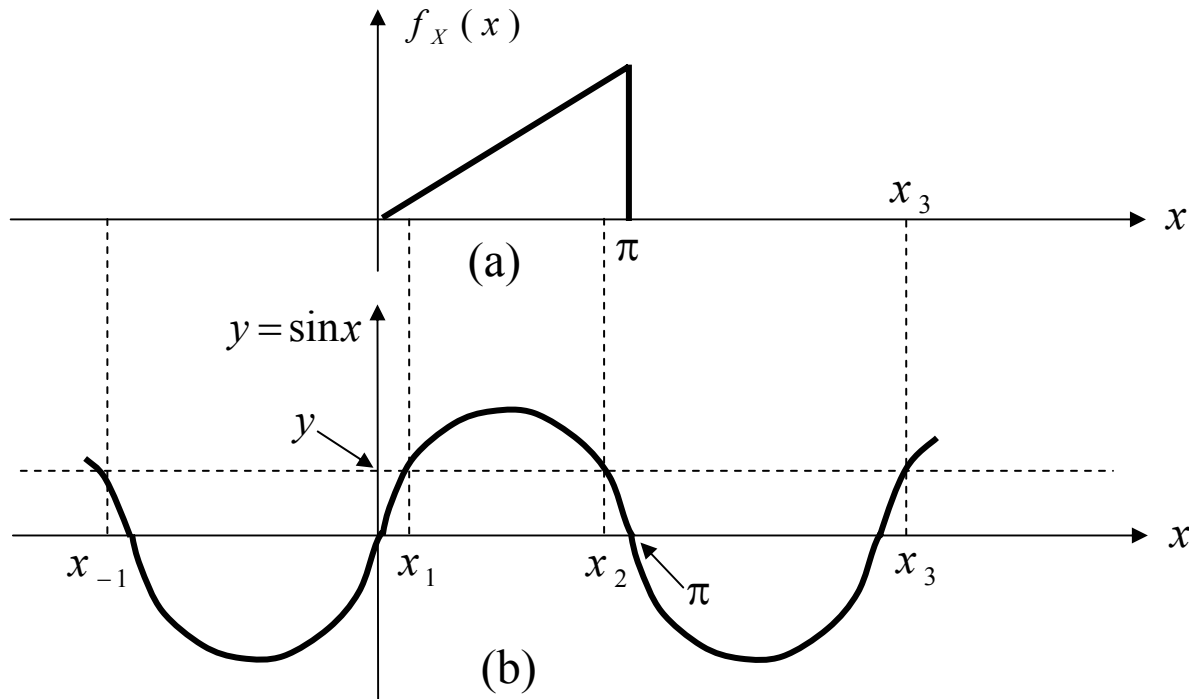


Fig. 5.6

Using this in (5-30), we obtain for $0 < y < 1$,

$$f_Y(y) = \sum_{\substack{i=-\infty \\ i \neq 0}}^{+\infty} \frac{1}{\sqrt{1-y^2}} f_X(x_i). \quad (5-36)$$

But from Fig. 5.6(a), in this case $f_X(x_{-1}) = f_X(x_3) = f_X(x_4) = \dots = 0$ (Except for $f_X(x_1)$ and $f_X(x_2)$ the rest are all zeros).

Thus (Fig. 5.7)

$$\begin{aligned}
 f_Y(y) &= \frac{1}{\sqrt{1-y^2}} (f_X(x_1) + f_X(x_2)) = \frac{1}{\sqrt{1-y^2}} \left(\frac{2x_1}{\pi^2} + \frac{2x_2}{\pi^2} \right) \\
 &= \frac{2(x_1 + \pi - x_1)}{\pi^2 \sqrt{1-y^2}} = \begin{cases} \frac{2}{\pi \sqrt{1-y^2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5-37)
 \end{aligned}$$

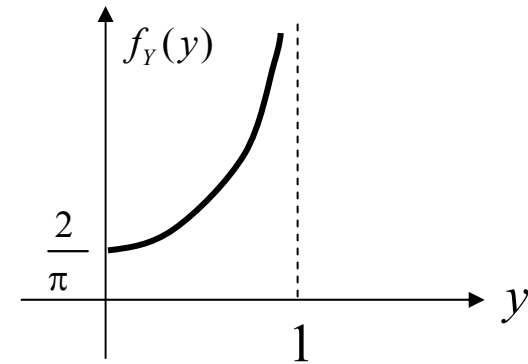


Fig. 5.7

Example 5.7: Let $Y = \tan X$ where $X \sim U(-\pi/2, \pi/2)$.

Determine $f_Y(y)$.

Solution: As x moves from $(-\pi/2, \pi/2)$, y moves from $(-\infty, +\infty)$.

From Fig.5.8(b), the function $Y = \tan X$ is one-to-one for $-\pi/2 < x < \pi/2$. For any y , $x_1 = \tan^{-1} y$ is the principal solution. Further

$$\frac{dy}{dx} = \frac{d \tan x}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

so that using (5-30)

$$f_Y(y) = \frac{1}{|dy/dx|_{x=x_1}} f_X(x_1) = \frac{1/\pi}{1+y^2}, \quad -\infty < y < +\infty, \quad (5-38)$$

which represents a Cauchy density function with parameter equal to unity (Fig. 5.9).

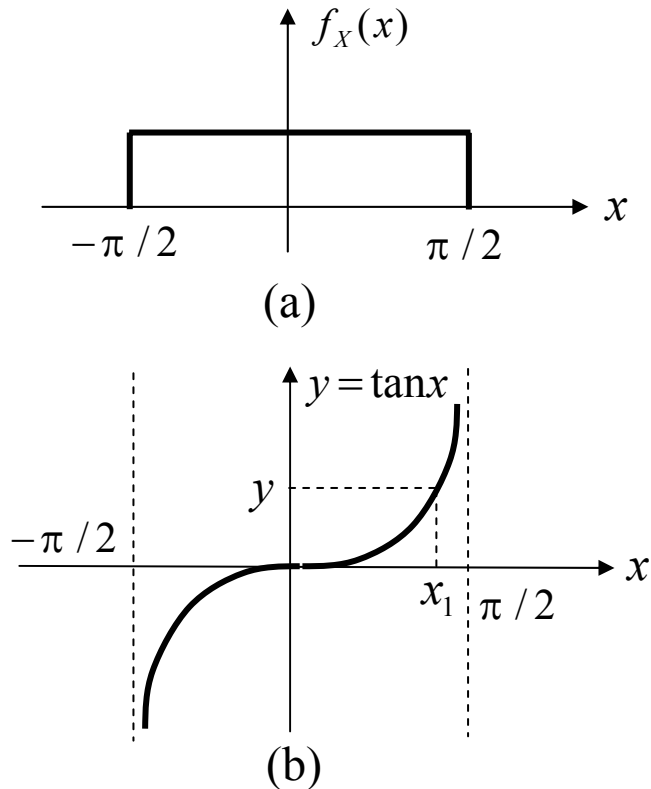


Fig. 5.8

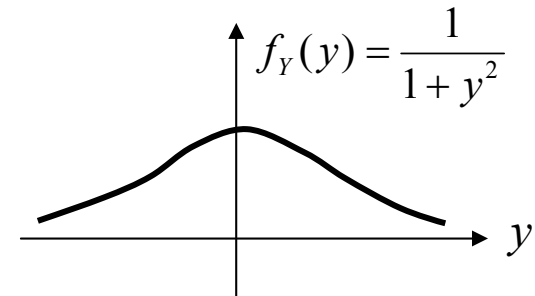


Fig. 5.9

Functions of a discrete-type r.v

Suppose X is a discrete-type r.v with

$$P(X = x_i) = p_i, \quad x = x_1, x_2, \dots, x_i, \dots \quad (5-39)$$

and $Y = g(X)$. Clearly Y is also of discrete-type, and when $x = x_i$, $y_i = g(x_i)$, and for those y_i

$$P(Y = y_i) = P(X = x_i) = p_i, \quad y = y_1, y_2, \dots, y_i, \dots \quad (5-40)$$

Example 5.8: Suppose $X \sim P(\lambda)$, so that

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \dots \quad (5-41)$$

Define $Y = X^2 + 1$. Find the p.m.f of Y .

Solution: X takes the values $0, 1, 2, \dots, k, \dots$ so that Y only takes the value $1, 2, 5, \dots, k^2 + 1, \dots$ and

$$P(Y = k^2 + 1) = P(X = k)$$

so that for $j = k^2 + 1$

$$P(Y = j) = P(X = \sqrt{j-1}) = e^{-\lambda} \frac{\lambda^{\sqrt{j-1}}}{(\sqrt{j-1})!}, \quad j = 1, 2, 5, \dots, k^2 + 1, \dots. \quad (5-42)$$