# 18. Power Spectrum

For a deterministic signal $x(t)$, the spectrum is well defined: If $X(\omega)$ represents its Fourier transform, i.e., if

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt, \qquad (18\text{-}1)$$

then $|X(\omega)|^2$ represents its energy spectrum. This follows from Parseval's theorem since the signal energy is given by

$$\int_{-\infty}^{+\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega = E. \qquad (18\text{-}2)$$

Thus $|X(\omega)|^2 \Delta\omega$ represents the signal energy in the band $(\omega, \omega + \Delta\omega)$ (see Fig 18.1).

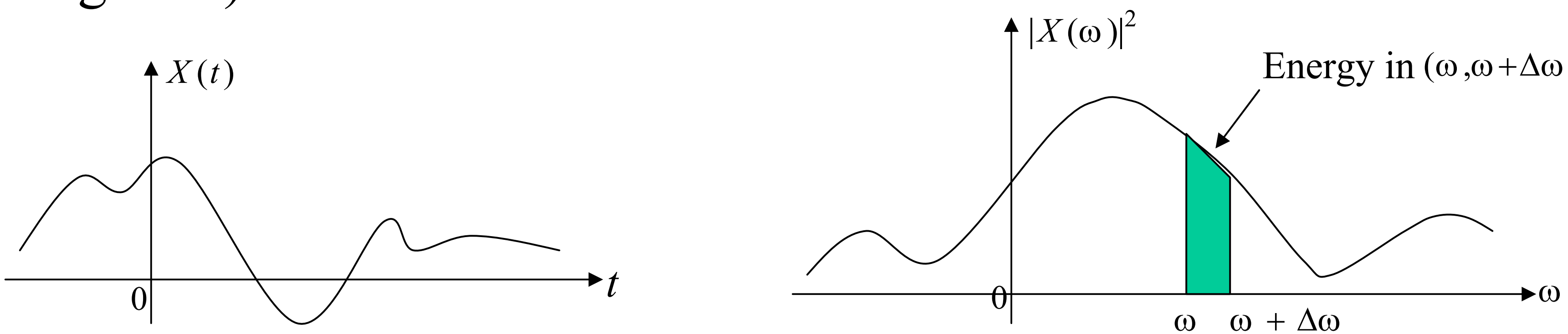



Fig 18.1

However for stochastic processes, a direct application of (18-1) generates a sequence of random variables for every $\omega$. Moreover, for a stochastic process, $E\{| X(t) |^2\}$ represents the ensemble average power (instantaneous energy) at the instant $t$.

To obtain the spectral distribution of power versus frequency for stochastic processes, it is best to avoid infinite intervals to begin with, and start with a finite interval $(-T, \ T)$ in (18-1). Formally, partial Fourier transform of a process $X(t)$ based on $(-T, \ T)$ is given by

$$X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt \qquad (18\text{-}3)$$

so that

$$\frac{| X_T(\omega) |^2}{2T} = \frac{1}{2T} \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2 \qquad (18\text{-}4)$$

represents the power distribution associated with that realization based on $(-T, \ T)$. Notice that (18-4) represents a random variable for every $\omega$, and its ensemble average gives, the average power distribution based on $(-T, \ T)$. Thus

$$P_T(\omega) = E\left\{\frac{|X_T(\omega)|^2}{2T}\right\} = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2 \qquad (18\text{-}5)$$

represents the power distribution of $X(t)$ based on $(-T, T)$. For wide sense stationary (w.s.s) processes, it is possible to further simplify (18-5). Thus if $X(t)$ is assumed to be w.s.s, then $R_{XX}(t_1,t_2) = R_{XX}(t_1 - t_2)$ and (18-5) simplifies to

$$P_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2.$$

Let $\tau = t_1 - t_2$ and proceeding as in (14-24), we get

$$P_T(\omega) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(2T - |\tau|)d\tau$$

$$= \int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}\left(1 - \tfrac{|\tau|}{2T}\right)d\tau \;\geq\; 0 \qquad (18\text{-}6)$$

to be the power distribution of the w.s.s. process $X(t)$ based on $(-T, T)$. Finally letting $T \to \infty$ in (18-6), we obtain

$$S_{XX}(\omega) = \lim_{T\to\infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}\,d\tau \geq 0 \tag{18-7}$$

to be the *power spectral density* of the w.s.s process $X(t)$. Notice that

$$R_{XX}(\omega) \xleftrightarrow{\text{F}\cdot\text{T}} S_{XX}(\omega) \geq 0. \tag{18-8}$$

i.e., the autocorrelation function and the power spectrum of a w.s.s Process form a Fourier transform pair, a relation known as the **Wiener-Khinchin Theorem**. From (18-8), the inverse formula gives

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega\tau}\,d\omega \tag{18-9}$$

and in particular for $\tau = 0,$ we get

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = R_{XX}(0) = E\{|X(t)|^2\} = P, \quad \textit{the total power.} \tag{18-10}$$

From (18-10), the area under $S_{XX}(\omega)$ represents the total power of the process $X(t)$, and hence $S_{XX}(\omega)$ truly represents the power spectrum. (Fig 18.2).

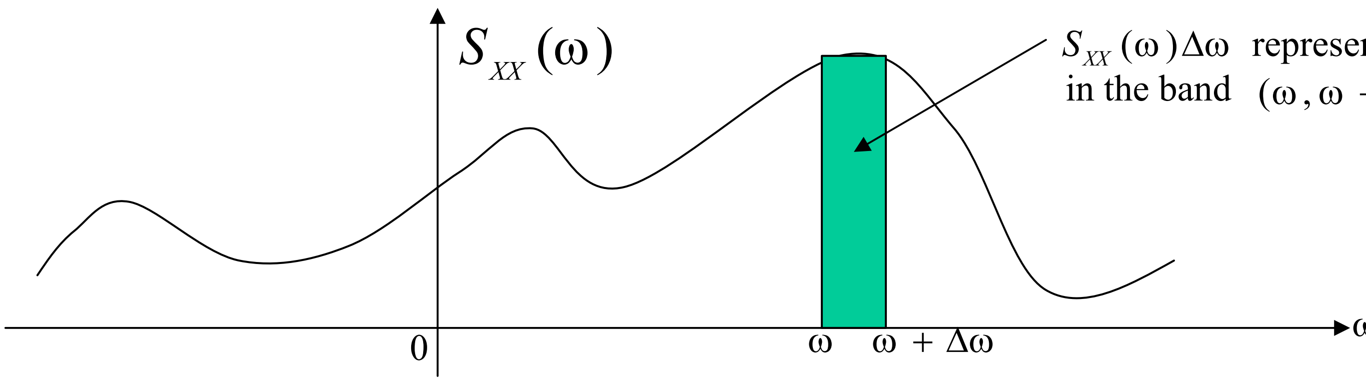



Fig 18.2

The nonnegative-definiteness property of the autocorrelation function in (14-8) translates into the "nonnegative" property for its Fourier transform (power spectrum), since from (14-8) and (18-9)

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i - t_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega(t_i - t_j)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) \left|\sum_{i=1}^{n} a_i e^{j\omega t_i}\right|^2 d\omega \geq 0. \qquad (18\text{-}11)$$

From (18-11), it follows that

$$R_{XX}(\tau)\ \textit{nonnegative-definite} \quad \Leftrightarrow \quad S_{XX}(\omega) \geq 0. \qquad (18\text{-}12)$$

If $X(t)$ is a real w.s.s process, then $R_{XX}(\tau) = R_{XX}(-\tau)$ so that

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{+\infty} R_{XX}(\tau) \cos\omega\tau \, d\tau$$

$$= 2\int_{0}^{\infty} R_{XX}(\tau) \cos\omega\tau \, d\tau = S_{XX}(-\omega) \geq 0 \qquad (18\text{-}13)$$

so that the power spectrum is an even function, (in addition to being real and nonnegative).

# Power Spectra and Linear Systems

If a w.s.s process $X(t)$ with autocorrelation function $R_{XX}(\tau) \;\leftrightarrow\; S_{XX}(\tau) \ge 0$ is applied to a linear system with impulse response $h(t)$, then the cross correlation function $R_{XY}(\tau)$ and the output autocorrelation function $R_{YY}(\tau)$ are given by (14-40)-(14-41). From there

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

Fig 18.3

$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau), \quad R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau). \qquad (18\text{-}14)$$

But if

$$f(t) \;\leftrightarrow\; F(\omega), \quad g(t) \;\leftrightarrow\; G(\omega) \qquad (18\text{-}15)$$

Then

$$f(t) * g(t) \;\leftrightarrow\; F(\omega)G(\omega) \qquad (18\text{-}16)$$

since

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} f(t) * g(t) e^{-j\omega t} dt$$

$$\mathcal{F}\{f(t)*g(t)\}=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau\right\}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega\tau}d\tau \int_{-\infty}^{+\infty} g(t-\tau)e^{-j\omega(t-\tau)}d(t-\tau)$$

$$=F(\omega)G(\omega). \qquad (18\text{-}17)$$

Using (18-15)-(18-17) in (18-14) we get

$$S_{XY}(\omega)=\mathcal{F}\{R_{XX}(\omega)*h^*(-\tau)\}=S_{XX}(\omega)H^*(\omega) \qquad (18\text{-}18)$$

since

$$\int_{-\infty}^{+\infty} h^*(-\tau)e^{-j\omega\tau}d\tau=\left(\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt\right)^*=H^*(\omega),$$

where

$$H(\omega)=\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt \qquad (18\text{-}19)$$

represents the transfer function of the system, and

$$S_{YY}(\omega)=\mathcal{F}\{R_{YY}(\tau)\}=S_{XY}(\omega)H(\omega)$$

$$=S_{XX}(\omega)\,|\,H(\omega)\,|^2. \qquad (18\text{-}20)$$

From (18-18), the cross spectrum need not be real or nonnegative; However the output power spectrum is real and nonnegative and is related to the input spectrum and the system transfer function as in (18-20). Eq. (18-20) can be used for system identification as well.

**W.S.S White Noise Process**: If $W(t)$ is a w.s.s white noise process, then from (14-43)

$$R_{WW}(\tau) = q\delta(\tau) \quad \Rightarrow \quad S_{WW}(\omega) = q. \qquad (18\text{-}21)$$

Thus the spectrum of a white noise process is flat, thus justifying its name. Notice that a white noise process is unrealizable since its total power is indeterminate.

From (18-20), if the input to an unknown system in Fig 18.3 is a white noise process, then the output spectrum is given by

$$S_{YY}(\omega) = q\,|\,H(\omega)\,|^2 \qquad (18\text{-}22)$$

Notice that the output spectrum captures the system transfer function characteristics entirely, and for rational systems Eq (18-22) may be used to determine the pole/zero locations of the underlying system.

**Example 18.1**: A w.s.s white noise process $W(t)$ is passed through a low pass filter (LPF) with bandwidth $B/2$. Find the autocorrelation function of the output process.

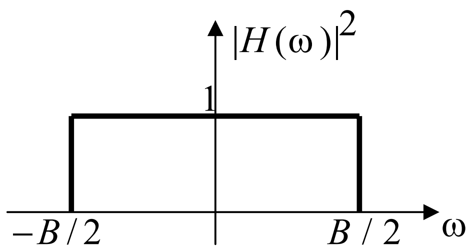
**Solution:** Let $X(t)$ represent the output of the LPF. Then from (18-22)

$$S_{XX}(\omega) = q\,|H(\omega)|^2 = \begin{cases} q, & |\omega| \le B/2 \\ 0, & |\omega| > B/2 \end{cases}. \tag{18-23}$$

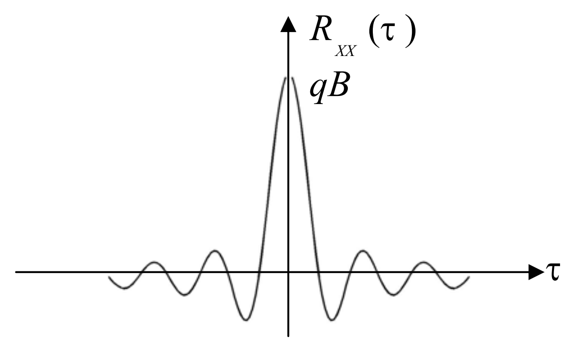
Inverse transform of $S_{XX}(\omega)$ gives the output autocorrelation function to be

$$R_{XX}(\tau) = \int_{-B/2}^{B/2} S_{XX}(\omega) e^{j\omega\tau}\, d\omega = q\int_{-B/2}^{B/2} e^{j\omega\tau}\, d\omega$$

$$= qB\,\frac{\sin(B\tau/2)}{(B\tau/2)} = qB\,\mathrm{sinc}(B\tau/2) \tag{18-24}$$




(a) LPF




(b)

Fig. 18.4

Eq (18-23) represents colored noise spectrum and (18-24) its autocorrelation function (see Fig 18.4).

**Example 18.2:** Let

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\tau)\,d\tau \qquad (18\text{-}25)$$

represent a "smoothing" operation using a moving window on the input process $X(t)$. Find the spectrum of the output $Y(t)$ in term of that of $X(t)$.

**Solution**: If we define an LTI system with impulse response $h(t)$ as in Fig 18.5, then in term of $h(t)$, Eq (18-25) reduces to

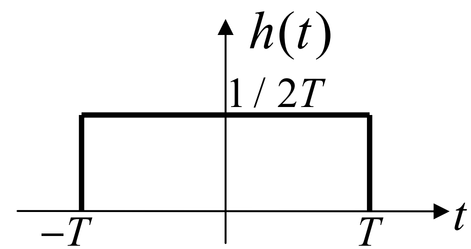


Fig 18.5

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)\,d\tau = h(t) * X(t) \qquad (18\text{-}26)$$

so that

$$S_{YY}(\omega) = S_{XX}(\omega)\,|\,H(\omega)\,|^2\,. \qquad (18\text{-}27)$$

Here

$$H(\omega) = \int_{-T}^{+T} \tfrac{1}{2T} e^{-j\omega t}\,dt = \operatorname{sinc}(\omega T) \qquad (18\text{-}28)$$

so that

$$S_{YY}(\omega) = S_{XX}(\omega)\,\text{sinc}^2(\omega T). \qquad (18\text{-}29)$$

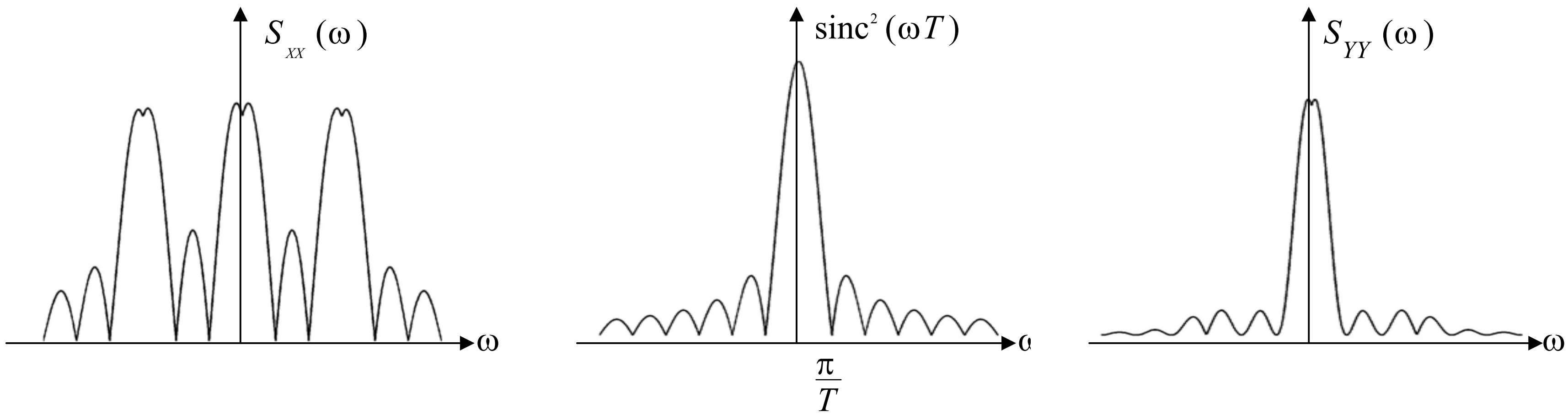



Fig 18.6

Notice that the effect of the smoothing operation in (18-25) is to suppress the high frequency components in the input (beyond $\pi / T$), and the equivalent linear system acts as a low-pass filter (continuous-time moving average) with bandwidth $2\pi / T$ in this case.

## Discrete – Time Processes

For discrete-time w.s.s stochastic processes $X(nT)$ with autocorrelation sequence $\{r_k\}_{-\infty}^{+\infty}$, (proceeding as above) or formally defining a continuous time process $X(t) = \sum_n X(nT)\delta(t - nT)$, we get the corresponding autocorrelation function to be

$$R_{XX}(\tau) = \sum_{k=-\infty}^{+\infty} r_k \delta(\tau - kT).$$

Its Fourier transform is given by

$$S_{XX}(\omega) = \sum_{k=-\infty}^{+\infty} r_k e^{-j\omega T} \geq 0, \qquad (18\text{-}30)$$

and it defines the power spectrum of the discrete-time process $X(nT)$. From (18-30),

$$S_{XX}(\omega) = S_{XX}(\omega + 2\pi / T) \qquad (18\text{-}31)$$

so that $S_{XX}(\omega)$ is a periodic function with period

$$2B = \frac{2\pi}{T}. \qquad (18\text{-}32)$$

This gives the inverse relation

$$r_k = \frac{1}{2B}\int_{-B}^{B} S_{XX}(\omega) e^{jk\omega T} d\omega \tag{18-33}$$

and

$$r_0 = E\{|X(nT)|^2\} = \frac{1}{2B}\int_{-B}^{B} S_{XX}(\omega) d\omega \tag{18-34}$$

represents the total power of the discrete-time process $X(nT)$. The input-output relations for discrete-time system $h(nT)$ in (14-65)-(14-67) translate into

$$S_{XY}(\omega) = S_{XX}(\omega) H^*(e^{j\omega}) \tag{18-35}$$

and

$$S_{YY}(\omega) = S_{XX}(\omega) \,|\, H(e^{j\omega}) \,|^2 \tag{18-36}$$

where

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(nT)\, e^{-j\omega nT} \tag{18-37}$$

represents the discrete-time system transfer function.

# Matched Filter

Let $r(t)$ represent a deterministic signal $s(t)$ corrupted by noise. Thus

$$r(t) = s(t) + w(t), \qquad 0 < t < t_0 \qquad (18\text{-}38)$$

where $r(t)$ represents the observed data, and it is passed through a receiver with impulse response $h(t)$. The output $y(t)$ is given by

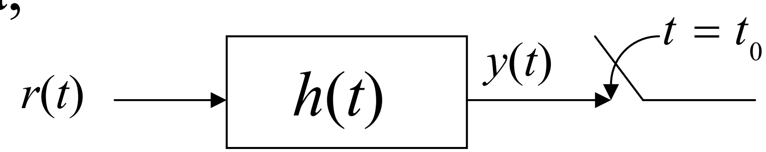



Fig 18.7 Matched Filter

$$y(t) \triangleq y_s(t) + n(t) \qquad (18\text{-}39)$$

where

$$y_s(t) = s(t) * h(t), \qquad n(t) = w(t) * h(t), \qquad (18\text{-}40)$$

and it can be used to make a decision about the presence of absence of $s(t)$ in $r(t)$. Towards this, one approach is to require that the receiver output signal to noise ratio $(SNR)_0$ at time instant $t_0$ be maximized. Notice that

$$(SNR)_0 \triangleq \frac{\text{Output signal power at } t=t_0}{\text{Average output noise power}} = \frac{|\,y_s(t_0)\,|^2}{E\{|\,n(t)\,|^2\}}$$

$$= \frac{|\,y_s(t_0)\,|^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{nn}(\omega)\,d\omega} = \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0}\,d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{WW}(\omega)\,|\,H(\omega)\,|^2\,d\omega} \qquad (18\text{-}41)$$

represents the output SNR, where we have made use of (18-20) to determine the average output noise power, and the problem is to maximize $(\mathrm{SNR})_0$ by optimally choosing the receiver filter $H(\omega)$.

**Optimum Receiver for White Noise Input:** The simplest input noise model assumes $w(t)$ to be white noise in (18-38) with spectral density $N_0$, so that (18-41) simplifies to

$$(SNR)_0 = \frac{\left|\int_{-\infty}^{+\infty} S(\omega)H(\omega)e^{j\omega t_0}\,d\omega\right|^2}{2\pi N_0 \int_{-\infty}^{+\infty} |\,H(\omega)\,|^2\,d\omega} \qquad (18\text{-}42)$$

and a direct application of Cauchy-Schwarz' inequality in (18-42) gives

$$(SNR)_0 \le \frac{1}{2\pi N_0}\int_{-\infty}^{+\infty} |S(\omega)|^2\, d\omega = \frac{\int_0^{+\infty} s(t)^2\, dt}{N_0} = \frac{E_s}{N_0} \qquad (18\text{-}43)$$

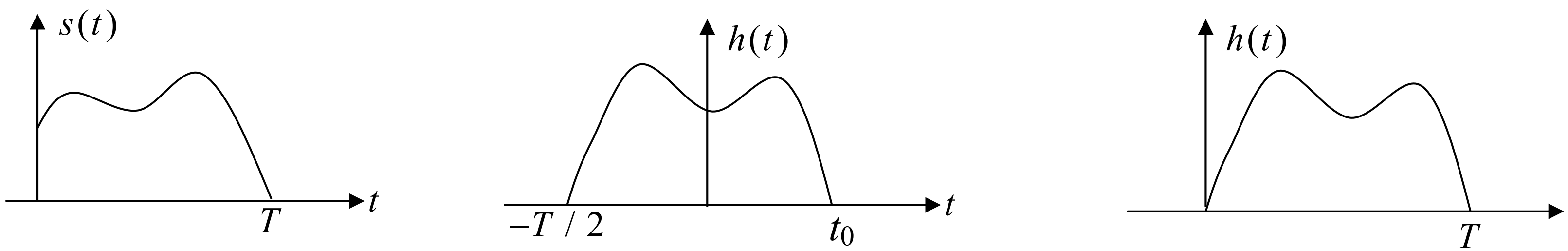
and equality in (18-43) is guaranteed if and only if

$$H(\omega) = S^*(\omega) e^{-j\omega t_0} \qquad (18\text{-}44)$$

or

$$h(t) = s(t_0 - t). \qquad (18\text{-}45)$$

From (18-45), the optimum receiver that maximizes the output SNR at $t = t_0$ is given by (18-44)-(18-45). Notice that (18-45) need not be causal, and the corresponding SNR is given by (18-43).




(a) (b) $t_0=T/2$ (c) $t_0=T$

Fig 18.8

Fig 18-8 shows the optimum $h(t)$ for two different values of $t_0$. In Fig 18.8 (b), the receiver is noncausal, whereas in Fig 18-8 (c) the receiver represents a causal waveform.

If the receiver is not causal, the optimum causal receiver can be shown to be

$$h_{opt}(t) = s(t_0 - t)u(t) \tag{18-46}$$

and the corresponding maximum $(SNR)_0$ in that case is given by

$$(SNR_0) = \frac{1}{N_0}\int_0^{t_0} s^2(t)dt \tag{18-47}$$

**Optimum Transmit Signal:** In practice, the signal $s(t)$ in (18-38) may be the output of a target that has been illuminated by a transmit signal $f(t)$ of finite duration $T$. In that case

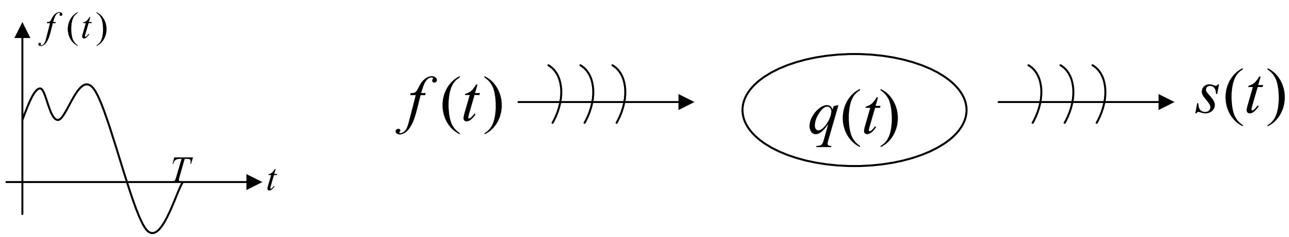



Fig 18.9

$$s(t) = f(t) * q(t) = \int_0^T f(\tau)q(t-\tau)d\tau, \tag{18-48}$$

where $q(t)$ represents the target impulse response. One interesting question in this context is to determine the optimum transmit signal $f(t)$ with normalized energy that maximizes the receiver output SNR at $t = t_0$ in Fig 18.7. Notice that for a given $s(t)$, Eq (18-45) represents the optimum receiver, and (18-43) gives the corresponding maximum $(\text{SNR})_0$. To maximize $(\text{SNR})_0$ in (18-43), we may substitute (18-48) into (18-43). This gives

$$(SNR)_0 = \int_0^\infty |\{\int_0^T q(t-\tau_1) f(\tau_1) d\tau_1\}|^2 \, dt$$

$$= \frac{1}{N_0} \int_0^T \int_0^T \underbrace{\int_0^\infty q(t-\tau_1) q^*(t-\tau_2) dt}_{\Omega(\tau_1,\tau_2)} f(\tau_2) d\tau_2 f(\tau_1) d\tau_1$$

$$= \frac{1}{N_0} \int_0^T \{\int_0^T \Omega(\tau_1,\tau_2) f(\tau_2) d\tau_2\} f(\tau_1) d\tau_1 \le \lambda_{\max} / N_0 \qquad (18\text{-}49)$$

where $\Omega(\tau_1,\tau_2)$ is given by

$$\Omega(\tau_1,\tau_2) = \int_0^\infty q(t-\tau_1) q^*(t-\tau_2) dt \qquad (18\text{-}50)$$

and $\lambda_{\max}$ is the largest eigenvalue of the integral equation

$$\int_0^T \Omega(\tau_1,\tau_2) f(\tau_2) d\tau_2 = \lambda_{\max} f(\tau_1), \qquad 0 < \tau_1 < T. \qquad (18\text{-}51)$$

and

$$\int_0^T f^2(t)dt = 1. \tag{18-52}$$

Observe that the kernal $\Omega(\tau_1,\tau_2)$ in (18-50) captures the target characteristics so as to maximize the output SNR at the observation instant, and the optimum transmit signal is the solution of the integral equation in (18-51) subject to the energy constraint in (18-52). Fig 18.10 show the optimum transmit signal and the companion receiver pair for a specific target with impulse response $q(t)$ as shown there .

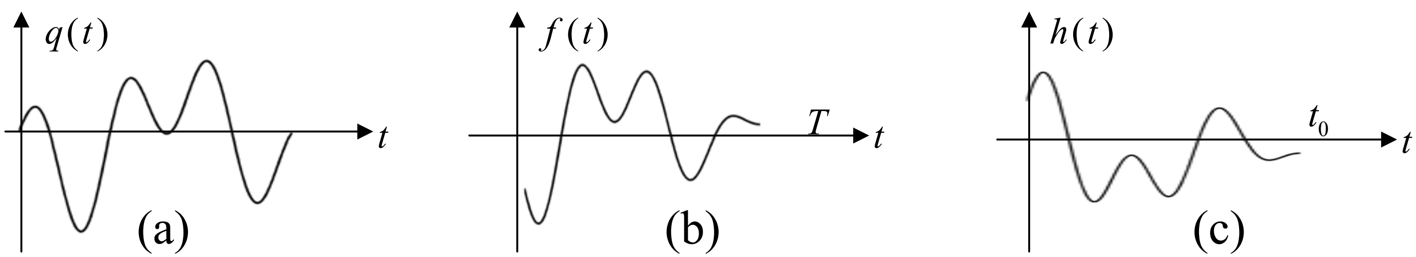



Fig 18.10

If the causal solution in (18-46)-(18-47) is chosen, in that case the kernel in (18-50) simplifies to

$$\Omega(\tau_1,\tau_2) = \int_0^{t_0} q(t-\tau_1)q^*(t-\tau_2)dt. \tag{18-53}$$

and the optimum transmit signal is given by (18-51). Notice that in the causal case, information beyond $t = t_0$ is not used.

What if the additive noise in (18-38) is not white?

Let $S_{ww}(\omega)$ represent a (non-flat) power spectral density. In that case, what is the optimum matched filter?

If the noise is *not* white, one approach is to *whiten* the input noise first by passing it through a whitening filter, and then proceed with the whitened output as before (Fig 18.7).

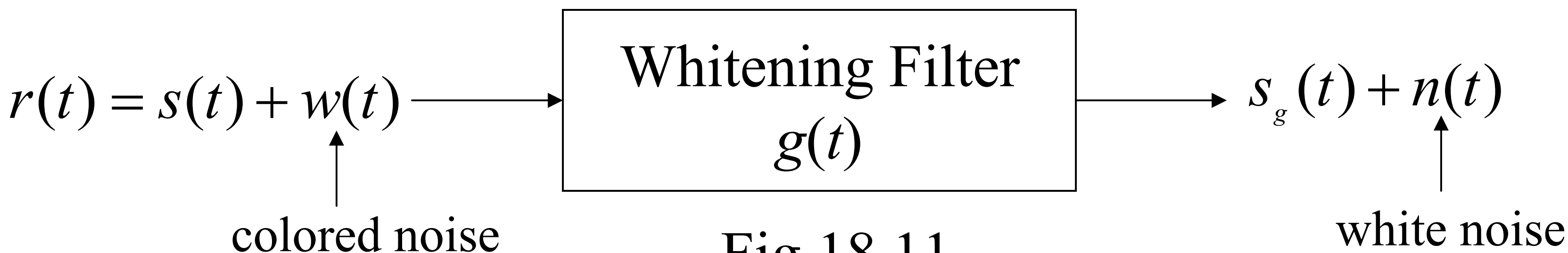



Fig 18.11

Notice that the signal part of the whitened output $s_g(t)$ equals

$$s_g(t) = s(t) * g(t) \qquad (18\text{-}54)$$

where $g(t)$ represents the whitening filter, and the output noise $n(t)$ is white with unit spectral density. This interesting idea due to

Wiener has been exploited in several other problems including prediction, filtering etc.

**Whitening Filter:** What is a whitening filter? From the discussion above, the output spectral density of the whitened noise process $S_{nn}(\omega)$ equals unity, since it represents the normalized white noise by design. But from (18-20)

$$1 = S_{nn}(\omega) = S_{WW}(\omega)\,|\,G(\omega)\,|^2,$$

which gives

$$|\,G(\omega)\,|^2 = \frac{1}{S_{WW}(\omega)}. \tag{18-55}$$

i.e., the whitening filter transfer function $G(\omega)$ satisfies the magnitude relationship in (18-55). To be useful in practice, it is desirable to have the whitening filter to be *stable* and *causal* as well. Moreover, at times its inverse transfer function also needs to be implementable so that it needs to be stable as well. How does one obtain such a filter (if any)? [See section 11.1 page 499-502, (and also page 423-424), Text for a discussion on obtaining the whitening filters.].

From there, any spectral density that satisfies the finite power constraint

$$\int_{-\infty}^{+\infty} S_{XX}(\omega)\,d\omega < \infty \tag{18-56}$$

and the Paley-Wiener constraint (see Eq. (11-4), Text)

$$\int_{-\infty}^{+\infty} \frac{|\log S_{XX}(\omega)|}{1+\omega^2}\,d\omega < \infty \tag{18-57}$$

can be factorized as

$$S_{XX}(\omega) = |H(j\omega)|^2 = H(s)H(-s)\big|_{s=j\omega} \tag{18-58}$$

where $H(s)$ together with its inverse function $1/H(s)$ represent two filters that are both analytic in Re $s > 0$. Thus $H(s)$ and its inverse $1/H(s)$ can be chosen to be s*table* and *causal* in (18-58). Such a filter is known as the Wiener factor, and since it has all its poles and zeros in the left half plane, it represents a *minimum phase factor*. In the rational case, if $X(t)$ represents a real process, then $S_{XX}(\omega)$ is even and hence (18-58) reads

$$0 \le S_{XX}(\omega^2) = \tilde{S}_{XX}(-s^2)\big|_{s=j\omega} = H(s)H(-s)\big|_{s=j\omega}\,. \qquad (18\text{-}59)$$

**Example 18.3:** Consider the spectrum

$$S_{XX}(\omega) = \frac{(\omega^2+1)(\omega^2-2)^2}{(\omega^4+1)}$$

which translates into

$$\tilde{S}_{XX}(-s^2) = \frac{(1-s^2)(2+s^2)^2}{1+s^4}.$$

The poles ($\times$) and zeros ($\circ$) of this function are shown in Fig 18.12. From there to maintain the symmetry condition in (18-59), we may group together the left half factors as

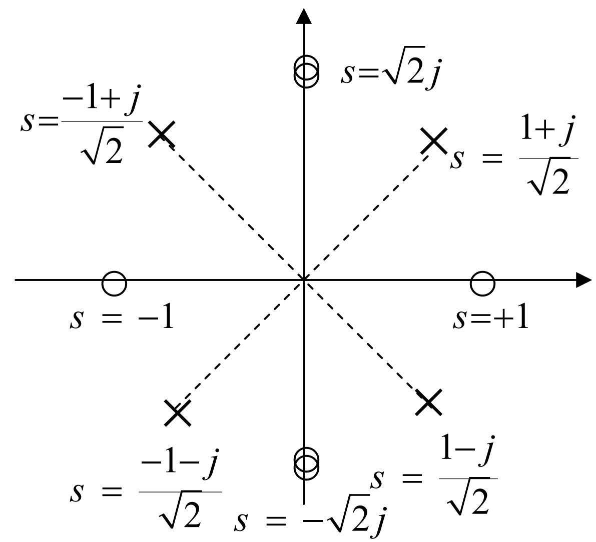



Fig 18.12

$$H(s) = \frac{(s+1)(s-\sqrt{2}j)(s+\sqrt{2}j)}{\left(s+\frac{1+j}{\sqrt{2}}\right)\left(s+\frac{1-j}{\sqrt{2}}\right)} = \frac{(s+1)(s^2+2)}{s^2+\sqrt{2}s+1}$$

and it represents the Wiener factor for the spectrum $S_{XX}(\omega)$ above. Observe that the poles and zeros (if any) on the $j\omega-$axis appear in even multiples in $S_{XX}(\omega)$ and hence half of them may be paired with $H$(s) (and the other half with $H(-s)$) to preserve the factorization condition in (18-58). Notice that $H(s)$ is stable, and so is its inverse.

More generally, if $H(s)$ is minimum phase, then ln $H(s)$ is analytic on the right half plane so that

$$H(\omega)=A(\omega)e^{-j\varphi(\omega)} \tag{18-60}$$

gives

$$\ln H(\omega)=\ln A(\omega)-j\varphi(\omega)\triangleq\int_0^{+\infty}b(t)e^{-j\omega t}dt.$$

Thus

$$\ln A(\omega)=\int_0^t b(t)\cos\omega t\ dt$$

$$\varphi(\omega)=\int_0^t b(t)\sin\omega t\ dt$$

and since $\cos\omega t$ and $\sin\omega t$ are Hilbert transform pairs, it follows that the phase function $\varphi(\omega)$ in (18-60) is given by the Hilbert transform of $\ln A(\omega)$. Thus

$$\varphi(\omega) = \mathcal{H}\{\ln A(\omega)\}. \tag{18-61}$$

Eq. (18-60) may be used to generate the unknown phase function of a minimum phase factor from its magnitude.

For discrete-time processes, the factorization conditions take the form (see (9-203)-(9-205), Text)

$$\int_{-\pi}^{\pi} S_{XX}(\omega)d\omega \ < \infty \tag{18-62}$$

and

$$\int_{-\pi}^{\pi} \ln S_{XX}(\omega)d\omega \ > -\infty. \tag{18-63}$$

In that case

$$S_{XX}(\omega) = |H(e^{j\omega})|^2$$

where the discrete-time system

$$H(z) = \sum_{k=0}^{\infty} h(k)z^{-k}$$

is analytic together with its inverse in $|z| > 1$. This unique minimum phase function represents the Wiener factor in the discrete-case.

**Matched Filter in Colored Noise:**

Returning back to the matched filter problem in colored noise, the design can be completed as shown in Fig 18.13.

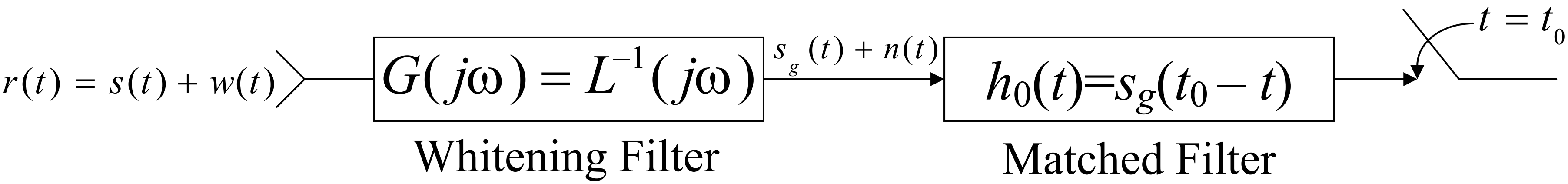



Fig 18.13

(Here $G(j\omega)$ represents the whitening filter associated with the noise spectral density $S_{WW}(\omega)$ as in (18-55)-(18-58). Notice that $G(s)$ is the inverse of the Wiener factor $L(s)$ corresponding to the spectrum $S_{WW}(\omega)$. i.e.,

$$L(s)L(-s)\,|_{s=j\omega} = |\,L(j\omega)\,|^2 = S_{WW}(\omega). \qquad (18\text{-}64)$$

The whitened output $s_g(\mathrm{t}) + n(t)$ in Fig 18.13 is similar to (18-38), and from (18-45) the optimum receiver is given by

$$h_0(t) = s_g(t_0 - t)$$

where

$$s_g(t) \leftrightarrow S_g(\omega) = G(j\omega)S(\omega) = L^{-1}(j\omega)S(\omega).$$

If we insist on obtaining the receiver transfer function $H(\omega)$ for the original colored noise problem, we can deduce it easily from Fig 18.14

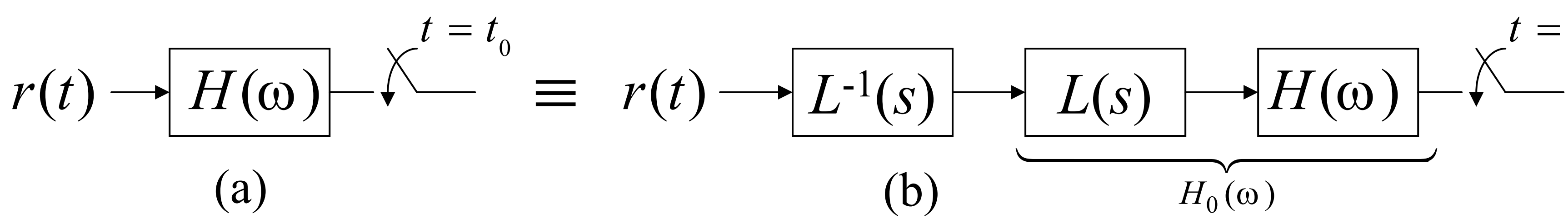



Fig 18.14

Notice that Fig 18.14 (a) and (b) are equivalent, and Fig 18.14 (b) is equivalent to Fig 18.13. Hence (see Fig 18.14 (b))

$$H_0(\omega) = L(j\omega)H(\omega)$$

or

$$H(\omega) = L^{-1}(j\omega)H_0(\omega) = L^{-1}(\omega)S_g^*(\omega)e^{-j\omega t_0}$$
$$= L^{-1}(\omega)\{L^{-1}(\omega)S(\omega)\}^* e^{-j\omega t_0} \qquad (18\text{-}65)$$

turns out to be the overall matched filter for the original problem. Once again, transmit signal design can be carried out in this case also.

## AM/FM Noise Analysis:

Consider the noisy AM signal

$$X(t) = m(t)\cos(\omega_0 t + \theta) + n(t), \qquad (18\text{-}66)$$

and the noisy FM signal

$$X(t) = A\cos(\omega_0 t + \varphi(t) + \theta) + n(t), \qquad (18\text{-}67)$$

where

$$\varphi(t) = \begin{cases} c\int_0^t m(\tau)d\tau & \text{FM} \\ cm(t) & \text{PM.} \end{cases} \qquad (18\text{-}68)$$

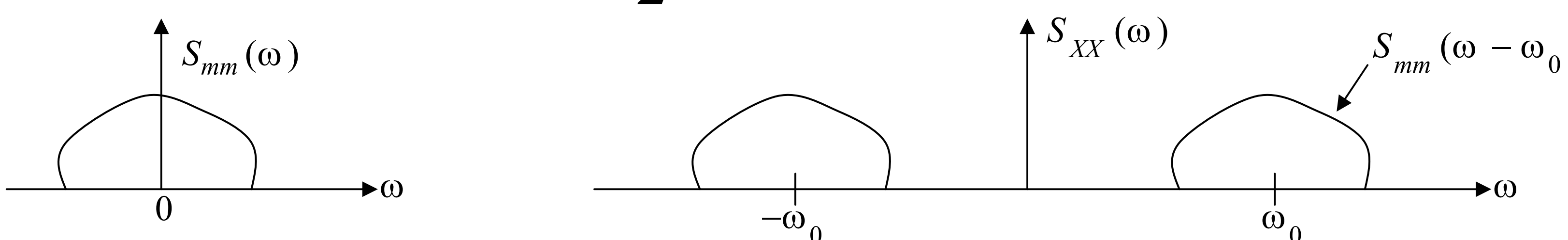
Here $m(t)$ represents the message signal and $\theta$ a random phase jitter in the received signal. In the case of FM, $\omega(t) = \varphi'(t) = c\, m(t)$ so that the instantaneous frequency is proportional to the message signal. We will assume that both the message process $m(t)$ and the noise process $n(t)$ are w.s.s with power spectra $S_{mm}(\omega)$ and $S_{nn}(\omega)$ respectively. We wish to determine whether the AM and FM signals are w.s.s, and if so their respective power spectral densities.

**Solution: AM signal:** In this case from (18-66), if we assume $\theta \sim U(0, 2\pi)$, then

$$R_{XX}(\tau) = \frac{1}{2} R_{mm}(\tau) \cos \omega_0 \tau + R_{nn}(\tau) \tag{18-69}$$

so that (see Fig 18.15)

$$S_{XX}(\omega) = \frac{S_{XX}(\omega - \omega_0) + S_{XX}(\omega + \omega_0)}{2} + S_{nn}(\omega). \tag{18-70}$$




(a) (b)

Fig 18.15

Thus AM represents a stationary process under the above conditions. What about FM?

**FM signal:** In this case (suppressing the additive noise component in (18-67)) we obtain

$$
\begin{aligned}
R_{XX}(t+\tau/2, t-\tau/2) &= A^2 E\{\cos(\omega_0(t+\tau/2)+\varphi(t+\tau/2)+\theta)\times \\
&\qquad \cos(\omega_0(t-\tau/2)+\varphi(t-\tau/2)+\theta)\} \\
&= \frac{A^2}{2} E\{\cos[\omega_0\tau+\varphi(t+\tau/2)-\varphi(t-\tau/2)] \\
&\qquad +\cos[2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2)+2\theta]\} \\
&= \frac{A^2}{2}[E\{\cos(\varphi(t+\tau/2)-\varphi(t-\tau/2))\}\cos\omega_0\tau \\
&\qquad -E\{\sin(\varphi(t+\tau/2)-\varphi(t-\tau/2))\}\sin\omega_0\tau]
\end{aligned}
\tag{18-71}
$$

since

$$
\begin{aligned}
&E\{\cos(2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2)+2\theta)\} \\
&= E\{\cos(2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2))\}\underbrace{E\{\cos 2\theta\}}_{0} \\
&-E\{\sin(2\omega_0 t+\varphi(t+\tau/2)+\varphi(t-\tau/2))\}\underbrace{E\{\sin 2\theta\}}_{0}=0.
\end{aligned}
$$

Eq (18-71) can be rewritten as

$$R_{XX}(t+\tau/2, t-\tau/2) = \frac{A^2}{2}[a(t,\tau)\cos\omega_0\tau - b(t,\tau)\sin\omega_0\tau] \tag{18-72}$$

where

$$a(t,\tau) \stackrel{\Delta}{=} E\{\cos(\varphi(t+\tau/2)-\varphi(t-\tau/2))\} \tag{18-73}$$

and

$$b(t,\tau) \stackrel{\Delta}{=} E\{\sin(\varphi(t+\tau/2)-\varphi(t-\tau/2))\} \tag{18-74}$$

In general $a(t,\tau)$ and $b(t,\tau)$ depend on both $t$ and $\tau$ so that noisy FM is *not* w.s.s in general, even if the message process $m(t)$ is w.s.s.
In the special case when m(t) is a stationary Gaussian process, from (18-68), $\varphi(t)$ is also a stationary Gaussian process with autocorrelation function

$$R_{\varphi'\varphi'}(\tau) = c^2 R_{mm}(\tau) = \frac{-d^2 R_{\varphi\varphi}(\tau)}{d\tau^2} \tag{18-75}$$

for the FM case. In that case the random variable

$$Y \stackrel{\Delta}{=} \varphi(t+\tau/2) - \varphi(t-\tau/2) \sim N(0, \sigma_Y^2) \tag{18-76}$$

where

$$\sigma_Y^2 = 2(R_{\varphi\varphi}(0) - R_{\varphi\varphi}(\tau)). \tag{18-77}$$

Hence its characteristic function is given by

$$E\{e^{j\omega Y}\} = e^{-\omega^2\sigma_Y^2/2} = e^{-(R_{\varphi\varphi}(0) - R_{\varphi\varphi}(\tau))\omega^2} \tag{18-78}$$

which for $\omega = 1$ gives

$$E\{e^{jY}\} = E\{\cos Y\} + jE\{\sin Y\} = a(t,\tau) + jb(t,\tau), \tag{18-79}$$

where we have made use of (18-76) and (18-73)-(18-74). On comparing (18-79) with (18-78) we get

$$a(t,\tau) = e^{-(R_{\varphi\varphi}(0) - R_{\varphi\varphi}(\tau))} \tag{18-80}$$

and

$$b(t,\tau) \equiv 0 \tag{18-81}$$

so that the FM autocorrelation function in (18-72) simplifies into

$$R_{XX}(\tau)=\frac{A^2}{2}e^{-(R_{\varphi\varphi}(0)-R_{\varphi\varphi}(\tau))}\cos\omega_0\tau. \tag{18-82}$$

Notice that for stationary Gaussian message input $m(t)$ (or $\varphi(t)$), the nonlinear output $X(t)$ is indeed strict sense stationary with autocorrelation function as in (18-82).

**Narrowband FM:** If $R_{\varphi\varphi}(0)\ll 1$, then (18-82) may be approximated as $(e^{-x}\simeq 1-x,\ \ |x| << 1)$

$$R_{XX}(\tau)=\frac{A^2}{2}\{(1-R_{\varphi\varphi}(0))+R_{\varphi\varphi}(\tau)\}\cos\omega_0\tau \tag{18-83}$$

which is similar to the AM case in (18-69). Hence narrowband FM and ordinary AM have equivalent performance in terms of noise suppression.

**Wideband FM:** This case corresponds to $R_{\varphi\varphi}(0)>1$. In that case a Taylor series expansion or $R_{\varphi\varphi}(\tau)$ gives

$$R_{\varphi\varphi}(\tau) = R_{\varphi\varphi}(0) + \frac{1}{2}R''_{\varphi\varphi}(0)\tau^2 + \cdots = R_{\varphi\varphi}(0) - \frac{c^2}{2}R_{mm}(0)\tau^2 + \cdots \quad (18\text{-}84)$$

and substituting this into (18-82) we get

$$R_{XX}(\tau) = \frac{A^2}{2}\left\{ e^{-\frac{c^2}{2}R_{mm}(0)\tau^2 + \cdots} \right\}\cos\omega_0\tau \qquad (18\text{-}85)$$

so that the power spectrum of FM in this case is given by

$$S_{XX}(\omega) = \tfrac{1}{2}\{S(\omega - \omega_0) + S(\omega + \omega_0)\} \qquad (18\text{-}86)$$

where

$$S(\omega) \simeq \frac{A^2}{2} e^{-\omega^2/2c^2 R_{mm}(0)}. \qquad (18\text{-}87)$$

Notice that $S_{XX}(\omega)$ always occupies infinite bandwidth irrespective of the actual message bandwidth (Fig 18.16)and this capacity to spread the message signal across the entire spectral band helps to reduce the noise effect in any band.

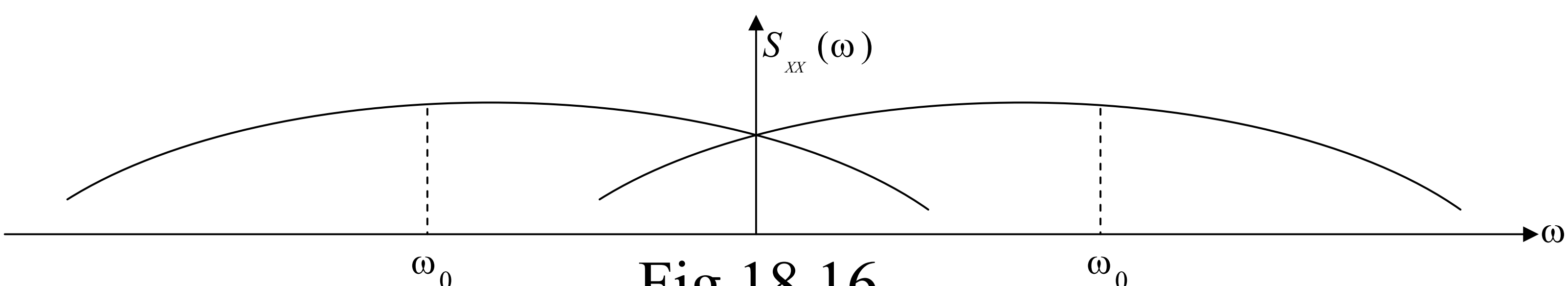



Fig 18.16

# Spectrum Estimation / Extension Problem

Given a finite set of autocorrelations $r_0, r_1, \cdots, r_n,$ one interesting problem is to extend the given sequence of autocorrelations such that the spectrum corresponding to the overall sequence is nonnegative for all frequencies. i.e., given $r_0, r_1, \cdots, r_n,$ we need to determine $r_{n+1}, r_{n+2}, \cdots$ such that

$$S(\omega) = \sum_{k=-\infty}^{\infty} r_k e^{-jk\omega} \geq 0. \qquad (18\text{-}88)$$

Notice that from (14-64), the given sequence satisfies $T_n > 0$, and at every step of the extension, this nonnegativity condition must be satisfied. Thus we must have

$$T_{n+k} > 0, \qquad k = 1,\ 2, \cdots. \qquad (18\text{-}89)$$

Let $r_{n+1} = x.$ Then

$$T_{n+1}=\left|\begin{array}{cc} & x \\ & r_n \\ T_n & \vdots \\ & r_1 \\ x^*,r_n^*,\ \cdots r_1^* & r_0 \end{array}\right|$$

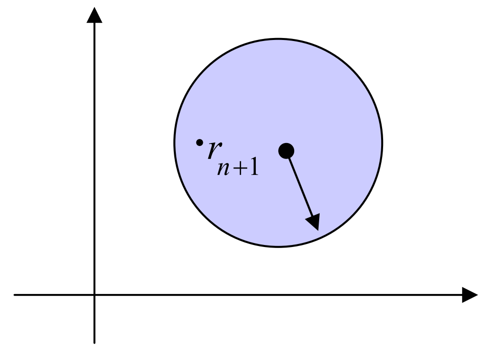



Fig 18.17

so that after some algebra

$$\Delta_{n+1}=\det T_{n+1}=\frac{\Delta_n^2-\Delta_{n-1}^2\,|\,x-\xi_n\,|^2}{\Delta_{n-1}}>0 \tag{18-90}$$

or

$$|\,r_{n+1}-\xi_n\,|^2 \quad < \quad \left(\frac{\Delta_n}{\Delta_{n-1}}\right)^2, \tag{18-91}$$

where

$$\xi_n \stackrel{\Delta}{=} \underline{a}^T T_{n-1}^{-1}\underline{b},\quad \underline{a}\stackrel{\Delta}{=}[r_1,r_2,\cdots,r_n]^T,\quad \underline{b}\stackrel{\Delta}{=}[r_n,r_{n-1},\cdots,r_1]^T. \tag{18-92}$$

Eq. (18-91) represents the interior of a circle with center $\xi_n$ and radius $\Delta_n / \Delta_{n-1}$ as in Fig 18.17, and geometrically it represents the admissible set of values for $r_{n+1}$. Repeating this procedure for $r_{n+2}, r_{n+3}, \cdots,$ it follows that the class of extensions that satisfy (18-85) are infinite.

It is possible to parameterically represent the class of all admissible spectra. Known as the trigonometric moment problem, extensive literature is available on this topic.
[See section 12.4 “Youla’s Parameterization”, pages 562-574, Text for a reasonably complete description and further insight into this topic.].