19. Series Representation of Stochastic Processes

Given information about a stochastic process $X(t)$ in $0 \le t \le T$, can this continuous information be represented in terms of a countable set of random variables whose relative importance decrease under some arrangement?

To appreciate this question it is best to start with the notion of a **Mean-Square periodic process**. A stochastic process $X(t)$ is said to be mean square (M.S) periodic, if for some $T > 0$

$$E[|X(t+T) - X(t)|^2] = 0 \quad \text{for all } t. \tag{19-1}$$

i.e $X(t) = X(t+T)$ with *probability* 1 for all t.
Suppose $X(t)$ is a W.S.S process. Then

$X(t)$ is mean-square perodic $\Leftrightarrow$ $R(\tau)$ is periodic in the ordinary sense, where $R(\tau) = E[X(t)X^*(t+T)]$

Proof: $(\Rightarrow)$ suppose $X(t)$ is M.S. periodic. Then

$$E[|X(t+T)-X(t)|^2]=0. \tag{19-2}$$

But from Schwarz' inequality

$$|E[X(t_1)\{X(t_2+T)-X(t_2)\}^*]|^2 \le E[|X(t_1)|^2]\underbrace{E[|X(t_2+T)-X(t_2)|^2]}_{0}$$

Thus the left side equals

$$E[X(t_1)\{X(t_2+T)-X(t_2)\}^*]=0$$

or

$$E[X(t_1)X^*(t_2+T)]=E[X(t_1)X^*(t_2)] \quad \Rightarrow \quad R(t_2-t_1+T)=R(t_2-t_1)$$

$$\Rightarrow \quad R(\tau+T)=R(\tau) \qquad \text{for any } \tau$$

i.e., $R(\tau)$ is periodic with period T. (19-3)

$(\Leftarrow)$ Suppose $R(\tau)$ is periodic. Then

$$E[|X(t+\tau)-X(t)|^2]=2R(0)-R(\tau)-R^*(\tau)=0$$

i.e., $X(t)$ is mean square periodic.

Thus if $X(t)$ is mean square periodic, then $R(\tau)$ is periodic and let

$$R(\tau) = \sum_{-\infty}^{+\infty} \gamma_n e^{jn\omega_0 \tau}, \qquad \omega_0 = \frac{2\pi}{T} \tag{19-4}$$

represent its Fourier series expansion. Here

$$\gamma_n = \frac{1}{T} \int_0^T R(\tau) e^{-jn\omega_0 \tau} d\tau. \tag{19-5}$$

In a similar manner define

$$c_k = \frac{1}{T} \int_0^T X(t) e^{jk\omega_0 t} dt \tag{19-6}$$

Notice that $c_k, \quad k = -\infty \to +\infty$ are random variables, and

$$E[c_k c_m^*] = \frac{1}{T^2} E[\int_0^T X(t_1) e^{jk\omega_0 t_1} dt_1 \int_0^T X^*(t_2) e^{-jm\omega_0 t_2} dt_2]$$

$$= \frac{1}{T^2} \int_0^T \int_0^T R(t_2 - t_1) e^{jk\omega_0 t_1} e^{-jm\omega_0 t_2} dt_1 dt_2$$

$$= \frac{1}{T} \int_0^T \underbrace{[\frac{1}{T} \int_0^T R \underbrace{(t_2 - t_1)}_{\tau} e^{-jm\omega_0 \overbrace{(t_2 - t_1)}^{\tau}} d \underbrace{(t_2 - t_1)}_{\tau}]}_{\gamma_m} e^{-j(m-k)\omega_0 t_1} dt_1$$

$$E[c_k c_m^*] = \gamma_m \underbrace{\{\tfrac{1}{T}\int_0^T e^{-j(m-k)\omega_0 t_1} dt_1\}}_{\delta_{m,k}} = \begin{cases} \gamma_m > 0, & k = m \\ 0 & k \neq m. \end{cases} \tag{19-7}$$

i.e., $\{c_n\}_{n=-\infty}^{n=+\infty}$ form a sequence of uncorrelated random variables, and, further, consider the partial sum

$$\tilde{X}_N(t) = \sum_{K=-N}^{N} c_k e^{-jk\omega_0 t}. \tag{19-8}$$

We shall show that $\tilde{X}_N(t) = X(t)$ in the mean square sense as $N \to \infty$. i.e.,

$$E[|X(t) - \tilde{X}_N(t)|^2] \to 0 \quad \text{as} \quad N \to \infty. \tag{19-9}$$

Proof:

$$E[|X(t) - \tilde{X}_N(t)|^2] = E[|X(t)|^2] - 2\,\mathrm{Re}[E(X^*(t)\tilde{X}_N(t)] + E[|\tilde{X}_N(t)|^2]. \tag{19-10}$$

But

$$E[|X(t)|^2] = R(0) = \sum_{k=-\infty}^{+\infty} \gamma_k,$$

and

$$E[X^*(t)\tilde{X}_N(t)] = E[\sum_{k=-N}^{N} c_k e^{-jk\omega_0 t} X^*(t)]$$

$$= \frac{1}{T}\sum_{k=-N}^{N} E[\int_0^T X(\alpha) e^{-jk\omega_0 (t-\alpha)} X^*(t) d\alpha]$$

$$= \sum_{k=-N}^{N} \underbrace{[\frac{1}{T}\int_0^T R(t-\alpha) e^{-jk\omega_0 (t-\alpha)} d(t-\alpha)]}_{\gamma_k} = \sum_{k=-N}^{N} \gamma_k. \quad (19\text{-}12)$$

Similarly

$$E[|\tilde{X}_N(t)|^2] = E[\sum_k \sum_m c_k c_m^* e^{j(k-m)\omega_0 t} = \sum_k \sum_m E[c_k c_m^*] e^{j(k-m)\omega_0 t} = \sum_{k=-N}^{N} \gamma_k.$$

$$\Rightarrow E[|X(t) - \tilde{X}_N(t)|^2] = 2(\sum_{k=-\infty}^{+\infty} \gamma_k - \sum_{k=-N}^{N} \gamma_k) \to 0 \quad \text{as} \quad N \to \infty \quad (19\text{-}13)$$

i.e.,

$$X(t) \doteq \sum_{k=-\infty}^{+\infty} c_k e^{-jk\omega_0 t}, \qquad -\infty < t < +\infty. \quad (19\text{-}14)$$

Thus mean square periodic processes can be represented in the form of a series as in (19-14). The stochastic information is contained in the random variables $c_k, \; k = -\infty \to +\infty$. Further these random variables are uncorrelated $(E\{c_k c_m^*\} = \gamma_k \delta_{k,m})$ and their variances $\gamma_k \to 0$ as $k \to \infty$. This follows by noticing that from (19-14)

$$\sum_{k=-\infty}^{+\infty} \gamma_k = R(0) = E[|X(t)|^2] = P < \infty.$$

Thus if the power P of the stochastic process is finite, then the positive sequence $\sum_{k=-\infty}^{+\infty} \gamma_k$ converges, and hence $\gamma_k \to 0$ as $k \to \infty$. This implies that the random variables in (19-14) are of relatively less importance as $k \to \infty$, and a finite approximation of the series in (19-14) is indeed meaningful.

The following natural question then arises: What about a general stochastic process, that is *not* mean square periodic? Can it be represented in a similar series fashion as in (19-14), if not in the whole interval $-\infty < t < \infty$, say in a finite support $0 \le t \le T$?

Suppose that it is indeed possible to do so for any arbitrary process $X(t)$ in terms of a certain sequence of orthonormal functions.

i.e.,

$$\tilde{X}(t) = \sum_{n=1}^{\infty} c_k \varphi_k(t) \tag{19-15}$$

where

$$c_k \triangleq \int_0^T X(t)\varphi_k^*(t)dt \tag{19-16}$$

$$\int_0^T \varphi_k(t)\varphi_n^*(t)dt = \delta_{k,n}, \tag{19-17}$$

and in the mean square sense

$$\tilde{X}(t) \doteq X(t) \quad \text{in} \quad 0 \le t \le T.$$

Further, as before, we would like the c_k s to be uncorrelated random variables. If that should be the case, then we must have

$$E[c_k c_m^*] = \lambda_m \delta_{k,m}. \tag{19-18}$$

Now

$$\begin{aligned} E[c_k c_m^*] &= E[\int_0^T X(t_1)\varphi_k^*(t_1)dt_1 \int_0^T X^*(t_2)\varphi_m(t_2)dt_2] \\ &= \int_0^T \varphi_k^*(t_1)\int_0^T E\{X(t_1)X^*(t_2)\}\ \varphi_m(t_2)dt_2 dt_1 \\ &= \int_0^T \varphi_k^*(t_1)\{\int_0^T R_{XX}(t_1,t_2)\ \varphi_m(t_2)dt_2\}dt_1 \qquad (19\text{-}19) \end{aligned}$$

and

$$\lambda_m \delta_{k,m} = \lambda_m \int_0^T \varphi_k^*(t_1)\varphi_m(t_1)dt_1. \tag{19-20}$$

Substituting (19-19) and (19-20) into (19-18), we get

$$\int_0^T \varphi_k^*(t_1)\{\int_0^T R_{XX}(t_1,t_2)\,\varphi_m(t_2)dt_2 - \lambda_m\varphi_m(t_1)\}dt_1 = 0. \tag{19-21}$$

Since (19-21) should be true for *every* $\varphi_k(t),\ \ k=1\to\infty,$ we must have

$$\int_0^T R_{XX}(t_1,t_2)\varphi_m(t_2)dt_2 - \lambda_m\varphi_m(t_1) \equiv 0,$$

or

$$\int_0^T R_{XX}(t_1,t_2)\varphi_m(t_2)dt_2 = \lambda_m\varphi_m(t_1), \quad 0<t_1<T, \quad m=1\to\infty. \tag{19-22}$$

i.e., the desired uncorrelated condition in (19-18) gets translated into the integral equation in (19-22) and it is known as the *Karhunen-Loeve* or K-L. integral equation. The functions $\{\varphi_k(t)\}_{k=1}^{\infty}$ are *not arbitrary* and they must be obtained by solving the integral equation in (19-22). They are known as the eigenvectors of the autocorrelation function of $R_{XX}(t_1, t_2)$. Similarly the set $\{\lambda_k\}_{k=1}^{\infty}$ represent the eigenvalues of the autocorrelation function. From (19-18), the eigenvalues λ_k represent the variances of the uncorrelated random variables c_k, $k = 1 \to \infty$. This also follows from Mercer's theorem which allows the representation

$$R_{XX}(t_1, t_2) = \sum_{k=1}^{\infty} \mu_k \phi_k(t_1) \phi_k^*(t_2), \quad 0 < t_1, t_2 < T, \tag{19-23}$$

where

$$\int_0^T \phi_k(t) \phi_m^*(t) dt = \delta_{k,m}.$$

Here $\phi_k(t)$ and μ_k, $k = 1 \to \infty$ are known as the eigenfunctions and eigenvalues of $R_{XX}(t_1, t_2)$ respectively. A direct substitution and simplification of (19-23) into (19-22) shows that

$$\varphi_k(t) = \phi_k(t), \quad \lambda_{\text{k}} = \mu_k, \quad k = 1 \to \infty. \tag{19-24}$$

Returning back to (19-15), once again the partial sum

$$\tilde{X}_N(t) = \sum_{k=1}^{N} c_k \varphi_k(t) \xrightarrow[N \to \infty]{} X(t), \quad 0 \le t \le T \tag{19-25}$$

in the mean square sense. To see this, consider

$$E[|X(t)-\tilde{X}_N(t)|^2] = E[|X(t)|^2] - E[X(t)\tilde{X}_N^*(t)] - E[X^*(t)\tilde{X}_N(t)] + E[|\tilde{X}_N(t)|^2]. \tag{19-26}$$

We have

$$E[|X(t)|^2] = R(t,t). \tag{19-27}$$

Also

$$\begin{aligned} E[X(t)\tilde{X}_N^*(t)] &= \sum_{k=1}^{N} X(t) c_k^* \varphi_k^*(t) \\ &= \sum_{k=1}^{N} \int_0^T E[X(t)X^*(\alpha)]\, \varphi_k^*(t)\varphi_k(\alpha)\, d\alpha \\ &= \sum_{k=1}^{N} \left(\int_0^T R(t,\alpha)\varphi_k(\alpha)\, d\alpha\right)\varphi_k^*(t) \\ &= \sum_{k=1}^{N} \lambda_k \varphi_k(t)\varphi_k^*(t) = \sum_{k=1}^{N} \lambda_k\, |\varphi_k(t)|^2. \end{aligned} \tag{19-28}$$

Similarly

$$E[X^*(t)\tilde{X}_N(t)] = \sum_{k=1}^{N} \lambda_k\, |\varphi_k(t)|^2 \tag{19-29}$$

and

$$E[|\tilde{X}_N(t)|^2] = \sum_k \sum_m E[c_k c_m^*] \varphi_k(t) \varphi_m^*(t) = \sum_{k=1}^{N} \lambda_k |\varphi_k(t)|^2. \qquad (19\text{-}30)$$

Hence (19-26) simplifies into

$$E[|X(t) - \widetilde{X}_N(t)|^2] = R(t,t) - \sum_{k=1}^{N} \lambda_k |\varphi_k(t)|^2 \to 0 \text{ as } \to \infty. \qquad (19\text{-}31)$$

i.e.,

$$X(t) \doteq \sum_{k=1}^{\infty} c_k \varphi_k(t), \qquad 0 \le t \le T, \qquad (19\text{-}32)$$

where the random variables $\{c_k\}_{k=1}^{\infty}$ are uncorrelated and faithfully represent the random process $X(t)$ in $0 \le t \le T$, provided $\varphi_k(t)$, $k = 1 \to \infty$, satisfy the K-L. integral equation.

Example 19.1: If $X(t)$ is a w.s.s white noise process, determine the sets $\{\varphi_k, \lambda_k\}_{k=1}^{\infty}$ in (19-22).

Solution: Here

$$R_{XX}(t_1, t_2) = q\delta(t_1 - t_2) \qquad (19\text{-}33)$$

and

$$\int_0^T R_{XX}(t_1,t_2)\varphi_k(t_2)\,dt_1 = q\int_0^T \delta(t_1-t_2)\varphi_k(t_2)\,dt_1$$

$$= q\,\varphi_k(t_1) \stackrel{\Delta}{=} \lambda_k\,\varphi_k(t_1) \qquad (19\text{-}34)$$

$\Rightarrow \varphi_k(t)$ can be arbitrary so long as they are orthonormal as in (19-17) and $\lambda_k = q, \quad k = 1 \to \infty.$ Then the power of the process

$$P = E[|X(t)|^2] = R(0) = \sum_{k=1}^{\infty}\lambda_k = \sum_{k=1}^{\infty} q = \infty$$

and in that sense white noise processes are *unrealizable*. However, if the received waveform is given by

$$r(t) = s(t) + n(t), \qquad 0 < t < T \qquad (19\text{-}35)$$

and $n(t)$ is a w.s.s white noise process, then since *any* set of orthonormal functions is sufficient for the white noise process representation, they can be chosen solely by considering the other signal $s(t)$. Thus, in (19-35)

$$R_{rr}(t_1-t_2) = R_{ss}(t_1-t_2) + q\delta(t_1-t_2) \qquad (19\text{-}36)$$

and if

$$R_{ss}(t_1 - t_2) = \sum_{k=1}^{\infty} \lambda_k \phi_k(t_1) \phi_k^*(t_2) \tag{19-37}$$

Then it follows that

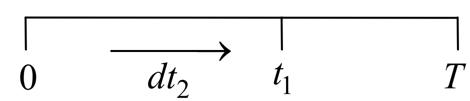
$$R_{rr}(t_1 - t_2) = \sum_{k=1}^{\infty} (\lambda_k + q) \phi_k(t_1) \phi_k^*(t_2). \tag{19-38}$$

Notice that the eigenvalues of $R_{ss}(t_1 - t_2)$ get incremented by q.

Example19.2: $X(t)$ is a Wiener process with

$$R_{XX}(t_1, t_2) = \alpha \min(t_1, t_2) = \begin{cases} \alpha t_2 & t_1 > t_2 \\ \alpha t_1 & t_1 \le t_2 \end{cases}, \qquad \alpha > 0 \tag{19-39}$$

In that case Eq. (19-22) simplifies to



$$\int_0^T R_{XX}(t_1, t_2) \varphi_k(t_2) dt_2 = \int_0^{t_1} R_{XX}(t_1, t_2) \varphi_k(t_2) dt_2 + \int_{t_1}^{T} R_{XX}(t_1, t_2) \varphi_k(t_2) dt_2 = \lambda_k \varphi_k(t_1),$$

and using (19-39) this simplifies to

$$\int_0^{t_1} \alpha t_2 \, \varphi_k(t_2) dt_2 + \int_{t_1}^{T} \alpha t_1 \, \varphi_k(t_2) dt_2 = \lambda_k \varphi_k(t_1). \tag{19-40}$$

Derivative with respect to t_1 gives [see Eqs. (8-5)-(8-6), Lecture 8]

$$\alpha\, t_1 \varphi_k(t_1) + (-1)\alpha\, t_1 \varphi_k(t_1) + \alpha \int_{t_1}^{T} \varphi_k(t_2)\,dt_2 = \lambda_k \dot{\varphi}_k(t_1)$$

or

$$\alpha \int_{t_1}^{T} \varphi_k(t_2)\,dt_2 = \lambda_k \dot{\varphi}_k(t_1). \tag{19-41}$$

Once again, taking derivative with respect to t_1, we obtain

$$\alpha(-1)\varphi_k(t_1) = \lambda_k \ddot{\varphi}_k(t_1)$$

or

$$\frac{d^2\varphi_k(t_1)}{dt_1^2} + \frac{\alpha}{\lambda_k}\varphi_k(t_1) = 0, \tag{19-42}$$

and its solution is given by

$$\varphi_k(t) = A_k \cos\sqrt{\tfrac{\alpha}{\lambda_k}}\,t + B_k \sin\sqrt{\tfrac{\alpha}{\lambda_k}}\,t.$$

But from (19-40)

$$\varphi_k(0) = 0, \tag{19-43}$$

and from (19-41)

$$\dot{\varphi}_k(T) = 0. \tag{19-44}$$

This gives

$$\begin{aligned} \varphi_k(0) &= A_k = 0, \qquad k = 1 \to \infty, \\ \dot{\varphi}_k(t) &= B_k \sqrt{\tfrac{\alpha}{\lambda_k}} \cos \sqrt{\tfrac{\alpha}{\lambda_k}}\, t, \end{aligned} \tag{19-45}$$

and using (19-44) we obtain

$$\dot{\varphi}_k(T) = B_k \sqrt{\tfrac{\alpha}{\lambda_k}} \cos \sqrt{\tfrac{\alpha}{\lambda_k}}\, T = 0$$

$$\Rightarrow \sqrt{\tfrac{\alpha}{\lambda_k}}\, T = (2k-1)\frac{\pi}{2} \tag{19-46}$$

$$\Rightarrow \quad \lambda_k = \frac{\alpha T^2}{(k - \frac{1}{2})^2 \pi^2}, \qquad k = 1 \to \infty. \tag{19-47}$$

Also

$$\varphi_k(t) = B_k \sin\sqrt{\tfrac{\alpha}{\lambda_k}}\; t, \qquad 0 \le t \le T. \tag{19-48}$$

Further, orthonormalization gives

$$\int_0^T \varphi_k^2(t)dt = B_k^2 \int_0^T \left(\sin\sqrt{\tfrac{\alpha}{\lambda_k}}\; t\right)^2 dt = B_k^2 \left[\int_0^T \left(\frac{1-\cos 2\sqrt{\frac{\alpha}{\lambda_k}}\; t}{2}\right) dt\right]$$

$$= B_k^2 \left(\frac{T}{2} - \frac{1}{2}\,\frac{\sin 2\sqrt{\frac{\alpha}{\lambda_k}}\; t}{2\sqrt{\frac{\alpha}{\lambda_k}}}\Bigg|_0^T\right) = B_k^2\left(\frac{T}{2} - \frac{\sin(2k-1)\pi - 0}{4\sqrt{\frac{\alpha}{\lambda_k}}}\right) = B_k^2\,\frac{T}{2} = 1$$

$$\Rightarrow B_k = \sqrt{2/T}.$$

Hence

$$\varphi_k(t) = \sqrt{\tfrac{2}{T}}\,\sin\left(\sqrt{\tfrac{\alpha}{\lambda_k}}\; t\right) = \sqrt{\tfrac{2}{T}}\,\sin\left(k - \tfrac{1}{2}\right)\frac{\pi t}{T}, \tag{19-49}$$

with λ_k as in (19-47) and c_k as in (19-16),

$$X(t)=\sum_{k=1}^{\infty} c_k \varphi_k(t)$$ is the desired series representation.

Example 19.3: Given

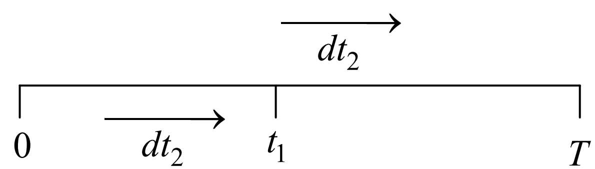
$$R_{XX}(\tau)=e^{-\alpha|\tau|}, \qquad \alpha>0, \tag{19-50}$$

find the orthonormal functions for the series representation of the underlying stochastic process $X(t)$ in $0<t<T$.

Solution: We need to solve the equation

$$\int_0^T e^{-\alpha|t_1-t_2|}\varphi_n(t_2)\,dt_2=\lambda_n\varphi_n(t_1). \tag{19-51}$$

Notice that (19-51) can be rewritten as,



$$\int_0^{t_1} e^{-\alpha\overbrace{(t_1-t_2)}^{\geq 0}}\varphi_n(t_2)\,dt_2+\int_{t_1}^T e^{-\alpha\overbrace{(t_2-t_1)}^{\geq 0}}\varphi_n(t_2)\,dt_2=\lambda_n\varphi_n(t_1) \tag{19-52}$$

Differentiating (19-52) once with respect to t_1, we obtain

$$\varphi_n(t_1) + \int_0^{t_1} (-\alpha) e^{-\alpha(t_1 - t_2)} \varphi_n(t_2) dt_2 - \varphi_n(t_1) + \int_{t_1}^{T} \alpha e^{-\alpha(t_2 - t_1)} \varphi_n(t_2) dt_2$$

$$= \lambda_n \frac{d\varphi_n(t_1)}{dt_1}$$

$$\Rightarrow \; -\int_0^{t_1} e^{-\alpha(t_1 - t_2)} \varphi_n(t_2) dt_2 + \int_{t_1}^{T} e^{-\alpha(t_2 - t_1)} \varphi_n(t_2) dt_2 = \frac{\lambda_n}{\alpha} \frac{d\varphi_n(t_1)}{dt_1} \qquad (19\text{-}53)$$

Differentiating (19-53) again with respect to t_1, we get

$$-\varphi_n(t_1) - \int_0^{t_1} (-\alpha) e^{-\alpha(t_1 - t_2)} \varphi_n(t_2) dt_2$$

$$-\varphi_n(t_1) + \int_{t_1}^{T} \alpha e^{-\alpha(t_2 - t_1)} \varphi_n(t_2) dt_2 = \frac{\lambda_n}{\alpha} \frac{d^2\varphi_n(t_1)}{dt_1^2}$$

or

$$-2\varphi_n(t_1) + \alpha \underbrace{\left[\int_0^{t_1} e^{-\alpha(t_1 - t_2)} \varphi_n(t_2) dt_2 + \int_{t_1}^{T} e^{-\alpha(t_2 - t_1)} \varphi_n(t_2) dt_2 \right]}_{\lambda_n \varphi_n(t_1) \quad \{\text{use (19-52)}\}}$$

$$= \frac{\lambda_n}{\alpha} \frac{d^2\varphi_n(t_1)}{dt_1^2}$$

or

$$(\alpha\lambda_n - 2)\varphi_n(t_1) = \frac{\lambda_n}{\alpha}\frac{d^2\varphi_n(t_1)}{dt_1^2}$$

or

$$\frac{d^2\varphi_n(t_1)}{dt_1^2} = \left(\frac{\alpha(\alpha\lambda_n - 2)}{\lambda_n}\right)\varphi_n(t_1). \tag{19-54}$$

Eq.(19-54) represents a second order differential equation. The solution for $\varphi_n(t)$ depends on the value of the constant $\alpha(\alpha\lambda_n - 2)/\lambda_n$ on the right side. We shall show that solutions exist in this case only if $\alpha\lambda_n < 2$, or

$$0 < \lambda_n < \frac{2}{\alpha}. \tag{19-55}$$

In that case $\alpha(\alpha\lambda_n - 2)/\lambda_n < 0.$

Let

$$\omega_n^2 \triangleq \frac{\alpha(2 - \alpha\lambda_n)}{\lambda_n} > 0, \tag{19-56}$$

and (19-54) simplifies to

$$\frac{d^2\varphi_n(t_1)}{dt_1^2} = -\omega_n^2\ \varphi_n(t_1). \tag{19-57}$$

General solution of (19-57) is given by

$$\varphi_n(t_1) = A_n \cos\omega_n t_1 + B_n \sin\omega_n t_1. \tag{19-58}$$

From (19-52)

$$\varphi_n(0) = \frac{1}{\lambda_n}\int_0^T e^{-\alpha t_2}\varphi_n(t_2)dt_2 \tag{19-59}$$

and

$$\varphi_n(T) = \frac{1}{\lambda_n}\int_0^T e^{-\alpha t_2}\varphi_n(T-t_2)dt_2. \tag{19-60}$$

Similarly from (19-53)

$$\dot{\varphi}_n(0) = \left.\frac{d\varphi_n(t_1)}{dt_1}\right|_{t_1=0} = \frac{\alpha}{\lambda_n}\int_0^T e^{-\alpha t_2}\varphi_n(t_2)dt_2 = \alpha\varphi_n(0) \tag{19-61}$$

and

$$\dot{\varphi}_n(T) = -\frac{\alpha}{\lambda_n}\int_0^T e^{-\alpha t_2}\varphi_n(T-t_2)dt_2 = -\alpha\varphi_n(T). \tag{19-62}$$

Using (19-58) in (19-61) gives

$$B_n\omega_n = \alpha A_n$$

or

$$\frac{A_n}{B_n} = \frac{\omega_n}{\alpha}, \tag{19-63}$$

and using (19-58) in (19-62), we have

$$-A_n\omega_n \sin\omega_n T + B_n\omega_n \cos\omega_n T = -\alpha(A_n \cos\omega_n T + B_n \sin\omega_n T),$$

$$\Rightarrow \quad (A_n\alpha + B_n\omega_n)\cos\omega_n T = (A_n\omega_n - B_n\alpha)\sin\omega_n T$$

or

$$\tan\omega_n T = \frac{2A_n\alpha}{A_n\omega_n - B_n\alpha} = \frac{2A_n\alpha / B_n\alpha}{\frac{A_n\omega_n}{B_n\alpha} - 1} = \frac{2A_n / B_n}{\frac{A_n}{B_n}\frac{\omega_n}{\alpha} - 1} = \frac{2(\omega_n/\alpha)}{(\frac{\omega_n}{\alpha})^2 - 1}.$$

Thus ω_n s are obtained as the solution of the transcendental equation

$$\tan\omega_n T = \frac{2(\omega_n/\alpha)}{(\omega_n/\alpha)^2 - 1}, \tag{19-64}$$

which simplifies to

$$\tan(\omega_n T/2) = -\frac{\omega_n}{\alpha}. \tag{19-65}$$

In terms of ω_n s from (19-56) we get

$$\lambda_n = \frac{2\alpha}{\alpha^2 + \omega_n^2} > 0. \tag{19-66}$$

Thus the eigenvalues are obtained as the solution of the transcendental equation (19-65). (see Fig 19.1). For each such λ_n (or ω_n^2), the corresponding eigenvector is given by (19-58). Thus

$$\begin{aligned}\varphi_n(t) &= A_n \cos\omega_n t + B_n \sin\omega_n t \\ &= c_n \sin(\omega_n t - \theta_n) = c_n \sin\omega_n (t - \tfrac{T}{2}), \qquad 0 < t < T\end{aligned} \tag{19-67}$$

since from (19-65)

$$\theta_n = \tan^{-1}\left(-\frac{A_n}{B_n}\right) = \tan^{-1}\left(-\frac{\omega_n}{\alpha}\right) = \omega_n T/2, \tag{19-68}$$

and c_n is a suitable normalization constant.

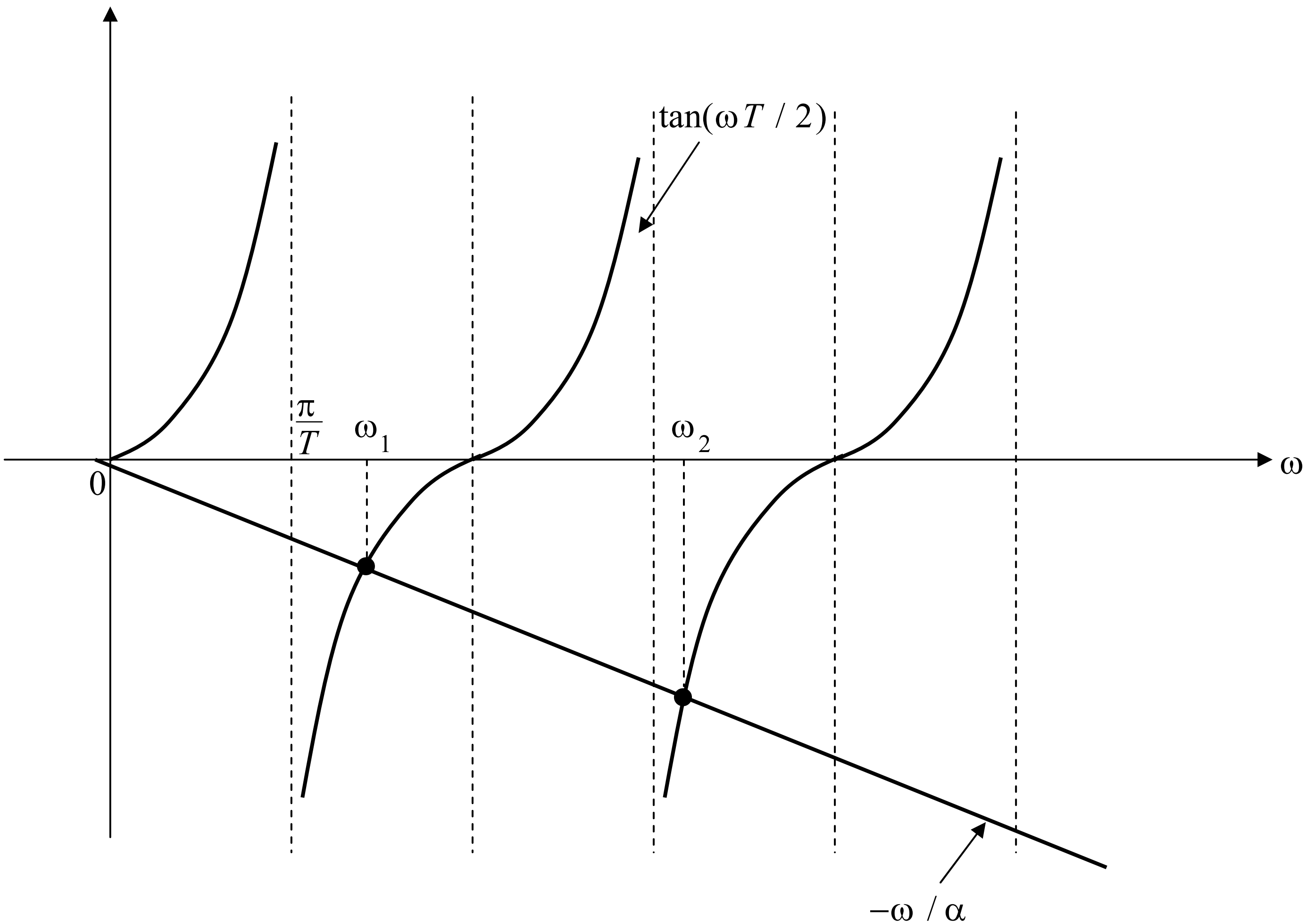


Fig 19.1 Solution for Eq.(19-65).

Karhunen – Loeve Expansion for Rational Spectra

[The following exposition is based on Youla's classic paper "The solution of a Homogeneous Wiener-Hopf Integral Equation occurring in the expansion of Second-order Stationary Random Functions," IRE Trans. on Information Theory, vol. 9, 1957, pp 187-193. Youla is tough. Here is a friendlier version. Even this may be skipped on a first reading. (Youla can be made only so much friendly.)]

Let $X(t)$ represent a w.s.s zero mean real stochastic process with autocorrelation function $R_{XX}(\tau) = R_{XX}(-\tau)$ so that its power spectrum

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega t} dt = 2\int_{0}^{\infty} R_{XX}(\tau) \cos\tau \, d\tau \qquad (19\text{-}69)$$

is nonnegative and an even function. If $S_{XX}(\omega)$ is rational, then the process $X(t)$ is said to be rational as well. $S_{XX}(\omega)$ rational and even implies

$$S_{XX}(\omega) = \frac{N(\omega^2)}{D(\omega^2)} \geq 0. \qquad (19\text{-}70)$$

The total power of the process is given by

$$P = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{N(\omega^2)}{D(\omega^2)}d\omega \tag{19-71}$$

and for P to be finite, we must have

(i) The degree $\delta(D) = 2n$ of the denominator polynomial $D(\omega^2)$ must exceed the degree $\delta(N) = 2m$ of the numerator polynomial $N(\omega^2)$ by at least two,

and

(ii) $D(\omega^2)$ must not have any zeros on the real-frequency $(s = j\omega)$ axis.

The s-plane $(s = \sigma + j\omega)$ extension of $S_{XX}(\omega)$ is given by

$$S_{XX}(\omega)\,|_{s=j\omega} \triangleq S(s^2) = \frac{N(-s^2)}{D(-s^2)}. \tag{19-72}$$

Thus

$$D(-s^2) = \prod_k (s^2 - \mu_k^2)^{k_i} \tag{19-73}$$

and the Laplace inverse transform of $(s^2 - \alpha^2)^{-k}$ is given by

$$\frac{1}{(s^2-\alpha^2)^k} \quad \leftrightarrow \quad \frac{(-1)^k}{(k-1)!} e^{-\alpha|\tau|} \sum_{j=1}^{k} \frac{(k+j-2)!}{(j-1)!(k-j)!} \frac{|\tau|^{k-j}}{(2\alpha)^{k+j-1}} \tag{19-74}$$

Let $\pm\mu_1, \ \pm\mu_2, \cdots, \ \pm\mu_n$ represent the roots of $D(-s^2)$. Then

$$0 < \operatorname{Re}\mu_1 \le \operatorname{Re}\mu_2 \le \cdots \le \operatorname{Re}\mu_n \tag{19-75}$$

Let $D^+(s)$ and $D^-(s)$ represent the left half plane (LHP) and the right half plane (RHP) products of these roots respectively. Thus

$$D(-s^2) = D^+(s)D^-(s), \tag{19-76}$$

where

$$D^+(s) = \prod_k (s+\mu_k)(s+\mu_k^*) = \sum_{k=0}^{n} d_k s^k = D^-(-s). \tag{19-77}$$

This gives

$$S(s^2) = \frac{N(-s^2)}{D(-s^2)} = \frac{C_1(s)}{D^+(s)} + \frac{C_2(s)}{D^-(s)} \tag{19-78}$$

Notice that $\frac{C_1(s)}{D^+(s)}$ has poles only on the LHP and its inverse (for all $t > 0$) converges only if the strip of convergence is to the right of *all* its poles. Similarly $C_2(s)\,/D^{-}(s)$ has poles only on the RHP and its inverse will converge only if the strip is to the left of all those poles. In that case, the inverse exists for $t<0$. In the case of $R_{XX}(\tau)$, from (19-78) its transform $N(s^2)\,/D(-s^2)$

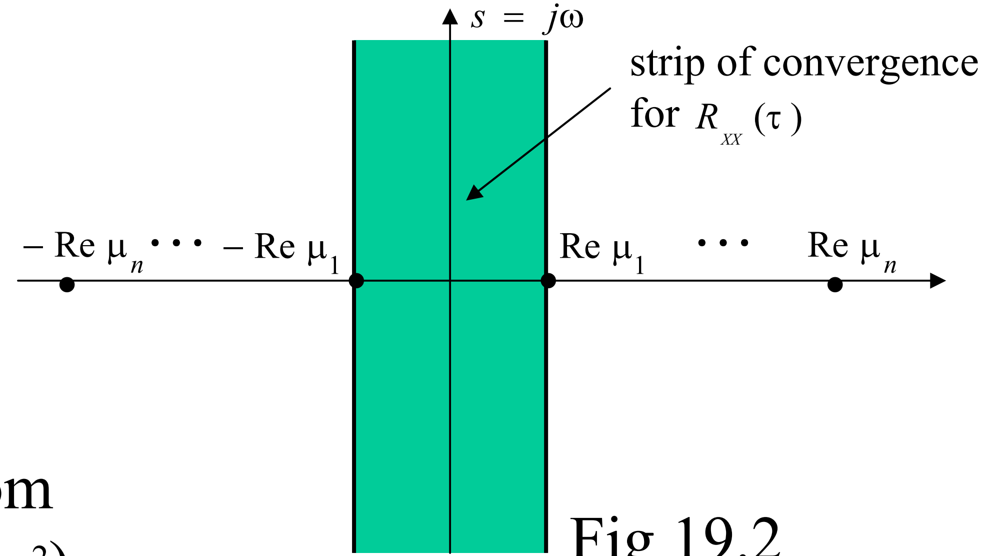


Fig 19.2

is defined only for $-\operatorname{Re}\mu_1 < \operatorname{Re} s < \operatorname{Re}\mu_1$ (see Fig 19.2). In particular, for $\tau > 0$, from the above discussion it follows that $R_{XX}(\tau)$ is given by the inverse transform of $C_1(s)\,/D^{+}(s)$. We need the solution to the integral equation

$$\varphi(t) = \lambda \int_0^T R_{XX}(t-\tau)\varphi(\tau)\,d\tau\,, \qquad 0<t<T \qquad (19\text{-}79)$$

that is valid *only* for $0<t<T$. (Notice that λ in (19-79) is the reciprocal of the eigenvalues in (19-22)). On the other hand, the right side (19-79) can be defined for every t. Thus, let

$$g(t) \triangleq \int_0^T R_{XX}(t-\tau)\varphi(\tau)\,d\tau\,, \qquad -\infty < t < +\infty \qquad (19\text{-}80)$$

and to confirm with the same limits, define

$$\phi(t) = \begin{cases} \varphi(t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}. \qquad (19\text{-}81)$$

This gives

$$g(t) = \int_{-\infty}^{+\infty} R_{XX}(t-\tau)\phi(\tau)\,d\tau \qquad (19\text{-}82)$$

and let

$$f(t) = \phi(t) - \lambda g(t) = \phi(t) - \lambda \int_{-\infty}^{+\infty} R_{XX}(t-\tau)\phi(\tau)\,d\tau. \qquad (19\text{-}83)$$

Clearly

$$f(t) = 0, \qquad 0 < t < T \qquad (19\text{-}84)$$

and for $t > T$

$$D^{+}\left(\tfrac{d}{dt}\right) f(t) = -\lambda \int_{-\infty}^{+\infty} \{\overbrace{D^{+}\left(\tfrac{d}{dt}\right) R_{XX}(t-\tau)}^{D^{+}(-\mu_k)=0}\}\phi(\tau)\,d\tau = 0, \qquad (19\text{-}85)$$

since $R_{XX}(t)$ is a sum of exponentials $\sum_k a_k e^{-\mu_k t}$, for $t > 0$. Hence it follows that for $t > T$, the function $f(t)$ must be a sum of exponentials $\sum_k a_k e^{-\mu_k t}$. Similarly for $t < 0$

$$D^-\left(\frac{d}{dt}\right) f(t) = -\lambda\int_{-\infty}^{+\infty} \{\overbrace{D^-\left(\frac{d}{dt}\right) R_{XX}(t-\tau)}^{D^-(\mu_k)=0}\}\phi(\tau)d\tau = 0,$$

and hence $f(t)$ must be a sum of exponentials $\sum_k b_k e^{\mu_k t}$, for $t<0$. Thus the overall Laplace transform of $f(t)$ has the form

$$F(s) = \underbrace{\frac{P(s)}{\underbrace{D^-(s)}_{\text{contributions in } t<0}}}_{} \underbrace{- e^{-sT}\frac{Q(s)}{\underbrace{D^+(s)}_{\text{contributes to } t>0}}}_{\text{contributions in } t>T} \tag{19-86}$$

where $P(s)$ and $Q(s)$ are polynomials of degree $n-1$ at most. Also from (19-83), the bilateral Laplace transform of $f(t)$ is given by

$$F(s) = \Phi(s)\left[1-\lambda\frac{N(-s^2)}{D(-s^2)}\right], \qquad -\operatorname{Re}\mu_1 < \operatorname{Re} s < \operatorname{Re}\mu_1 \tag{19-87}$$

Equating (19-86) and (19-87) and simplifying, Youla obtains the key identity

$$\Phi(s) = \frac{P(s)D^+(s) - e^{-sT}Q(s)D^-(s)}{D(-s^2)-\lambda N(-s^2)}. \tag{19-88}$$

Youla argues as follows: The function $\Phi(s)=\int_0^T \phi(t)e^{-st}dt$ is an entire function of s, and hence it is free of *poles* on the *entire* finite s-plane $(-\infty < \text{Re}\, s < +\infty)$. However, the denominator on the right side of (19-88) is a polynomial and its roots contribute to poles of $\Phi(s)$. Hence all such poles must be cancelled by the numerator. As a result the numerator of $\Phi(s)$ in (19-88) must possess exactly the *same set* of zeros as its denominator to the respective order at least.

Let $\pm\omega_1(\lambda),\ \pm\omega_2(\lambda), \cdots,\ \pm\omega_n(\lambda)$ be the (distinct) zeros of the denominator polynomial $D(-s^2) - \lambda N(-s^2)$. Here we assume that λ is an eigenvalue for which all ω_k's are distinct. We have

$$0 < \text{Re}\,\omega_1(\lambda) < \text{Re}\,\omega_2(\lambda) < \cdots < \text{Re}\,\omega_n(\lambda) < \infty. \qquad (19\text{-}89)$$

These ω_k's also represent the zeros of the numerator polynomial $P(s)D^+(s) - e^{-sT}Q(s)D^-(s)$. Hence

$$D^+(\omega_k)P(\omega_k) = e^{-\omega_k T} D^-(\omega_k)Q(\omega_k) \qquad (19\text{-}90)$$

and

$$D^+(-\omega_k)P(-\omega_k) = e^{\omega_k T} D^-(-\omega_k)Q(-\omega_k) \qquad (19\text{-}91)$$

which simplifies into

$$D^-(\omega_k)P(-\omega_k) = e^{\omega_k T} D^+(\omega_k)Q(-\omega_k). \qquad (19\text{-}92)$$

From (19-90) and (19-92) we get

$$P(\omega_k)P(-\omega_k) = Q(\omega_k)Q(-\omega_k), \qquad k = 1, 2, \cdots, n \tag{19-93}$$

i.e., the polynomial

$$L(s) = P(s)P(-s) - Q(s)Q(-s) \tag{19-94}$$

which is at most of degree $n - 1$ in s^2 vanishes at $\omega_1^2, \omega_2^2, \cdots, \omega_n^2$ (for n distinct values of s^2). Hence

$$L(s^2) \equiv 0 \tag{19-95}$$

or

$$P(s)P(-s) = Q(s)Q(-s). \tag{19-96}$$

Using the linear relationship among the coefficients of $P(s)$ and $Q(s)$ in (19-90)-(19-91) it follows that

$$P(s) = \pm Q(s) \qquad \text{or} \qquad P(s) = \pm Q(-s) \tag{19-97}$$

are the only solutions that are consistent with each of those equations, and together we obtain

$$P(s) = \pm Q(-s) \tag{19-98}$$

as the *only solution* satisfying both (19-90) and (19-91). Let

$$P(s) = \sum_{i=0}^{n-1} p_i \, s^i . \tag{19-99}$$

In that case (19-90)-(19-91) simplify to (use (19-98))

$$\begin{aligned} & P(\omega_k) D^+(\omega_k) \mp e^{-\omega_k T} D^-(\omega_k) P(-\omega_k) \\ & = \sum_{i=0}^{n-1} \{1 \mp (-1)^i a_k\} \omega_k^i \, p_i = 0, \qquad k = 1, 2, \cdots, n \end{aligned} \tag{19-100}$$

where

$$a_k = \frac{D^-(\omega_k)}{D^+(\omega_k)} e^{-\omega_k T} = \frac{D^+(-\omega_k)}{D^+(\omega_k)} e^{-\omega_k T} . \tag{19-101}$$

For a nontrivial solution to exist for $p_0, p_1, \cdots, p_{n-1}$ in (19-100), we must have

$$
\Delta_{1,2} = \begin{vmatrix} (1 \mp a_1) & (1 \pm a_1)\omega_1 & \cdots & (1 \mp (-1)^{n-1} a_1)\omega_1^{n-1} \\ (1 \mp a_2) & (1 \pm a_2)\omega_2 & \cdots & (1 \mp (-1)^{n-1} a_2)\omega_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ (1 \mp a_n) & (1 \pm a_n)\omega_n & \cdots & (1 \mp (-1)^{n-1} a_n)\omega_n^{n-1} \end{vmatrix} = 0. \quad (19\text{-}102)
$$

The two determinant conditions in (19-102) must be solved together to obtain the eigenvalues $\lambda_i's$ that are implicitly contained in the $a_i's$ and $\omega_i's$ (Easily said than done!).

To further simplify (19-102), one may express a_k in (19-101) as

$$
a_k = e^{-2\theta_k}, \qquad k = 1,\ 2,\ \cdots, n \qquad (19\text{-}103)
$$

so that

$$
\tanh\theta_k = \frac{e^{\theta_k} - e^{-\theta_k}}{e^{\theta_k} + e^{-\theta_k}} = \frac{1 - a_k}{1 + a_k} = \frac{D^+(\omega_k) - e^{-\omega_k T} D^+(-\omega_k)}{D^+(\omega_k) + e^{-\omega_k T} D^+(-\omega_k)}
$$

$$
= \frac{e^{\omega_k T/2} D^+(\omega_k) - e^{-\omega_k T/2} D^+(-\omega_k)}{e^{\omega_k T/2} D^+(\omega_k) + e^{-\omega_k T/2} D^+(-\omega_k)} \qquad (19\text{-}104)
$$

Let

$$D^{+}(s) = d_0 + d_1 s + \cdots + d_n s^n \tag{19-105}$$

and substituting these known coefficients into (19-104) and simplifying we get

$$\tanh\,\theta_k = \frac{(d_0 + d_2\omega_k^2 + \cdots)\tanh\,(\omega_k T/2) + (d_1\omega_k + d_3\omega_k^3 + \cdots)}{(d_0 + d_2\omega_k^2 + \cdots) + (d_1\omega_k + d_3\omega_k^3 + \cdots)\tanh\,(\omega_k T/2)} \tag{19-106}$$

and in terms of $\tanh\theta_k$, Δ_2 in (19-102) simplifies to

$$\begin{vmatrix} 1 & \omega_1 \tanh\theta_1 & \omega_1^2 & \omega_1^3 \tanh\theta_1 & \cdots & \omega_1^{n-1}\tanh\theta_1 \\ 1 & \omega_2 \tanh\theta_2 & \omega_2^2 & \omega_2^3 \tanh\theta_2 & \cdots & \omega_2^{n-1}\tanh\theta_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n \tanh\theta_n & \omega_n^2 & \omega_n^3 \tanh\theta_n & \cdots & \omega_n^{n-1}\tanh\theta_n \end{vmatrix} = 0 \tag{19-107}$$

if n is even (if n is odd the last column in (19-107) is simply $[\omega_1^{n-1},\ \omega_2^{n-1}, \cdots,\ \omega_n^{n-1}]^T$). Similarly Δ_1 in (19-102) can be obtained by replacing $\tanh\theta_k$ with $\coth\theta_k$ in (19-107).

To summarize determine the roots $\omega_k\,'s$ with $\mathrm{Re}(\omega_i) > 0$ that satisfy

$$D(-\omega_k^2) - \lambda N(-\omega_k^2) = 0, \qquad k = 1,\ 2,\ \cdots\ ,n \tag{19-108}$$

in terms of λ, and for every such ω_k, determine θ_k using (19-106). Finally using these $\omega_k\ s$ and $\tanh\ \theta_k\ s$ in (19-107) and its companion equation Δ_1, the eigenvalues $\lambda_k\ s$ are determined. Once $\lambda_k\ s$ are obtained, $p_k\ s$ can be solved using (19-100), and using that $\Phi_i(s)$ can be obtained from (19-88).

Thus

$$\Phi_i(s) = \frac{D^+(s)P(s,\lambda_i) - e^{-sT}D^-(s)Q(s,\lambda_i)}{D(-s^2) - \lambda_i N(-s^2)} \tag{19-109}$$

and

$$\phi_i(t) = L^{-1}\{\Phi_i(s)\}. \tag{19-110}$$

Since $\Phi_i(s)$ is an entire function in (19-110), the inverse Laplace transform in (19-109) can be performed through *any* strip of convergence in the s-plane, and in particular if we use the strip

$\operatorname{Re} s > \operatorname{Re}(\omega_n)$ (to the right of all $\operatorname{Re}(\omega_i)$), then the two inverses

$$L^{-1}\left\{\frac{D^+(s)P(s)}{D(-s^2)-\lambda N(-s^2)}\right\}, \qquad L^{-1}\left\{\frac{D^-(s)Q(s)}{D(-s^2)-\lambda N(-s^2)}\right\} \tag{19-111}$$

obtained from (19-109) will be causal. As a result $L^{-1}\left\{e^{-sT}\frac{D^-(s)Q(s)}{D(-s^2)-\lambda N(-s^2)}\right\}$

will be nonzero only for $t > T$ and using this in (19-109)-(19-110) we conclude that $\phi_i(t)$ for $0 < t < T$ has contributions only from the first term in (19-111). Together with (19-81), finally we obtain the desired eigenfunctions to be

$$\varphi_k(t) = L^{-1}\left\{\frac{D^+(s)P(s,\lambda_k)}{D(-s^2)-\lambda_k N(-s^2)}\right\}, \quad 0 < t < T, \tag{19-112}$$

$$\operatorname{Re} s > \operatorname{Re}\omega_n > 0, \quad k = 1, 2, \cdots, n$$

that are orthogonal by design. Notice that in general (19-112) corresponds to a sum of modulated exponentials.

Next, we shall illustrate this procedure through some examples. First, we shall re-do Example 19.3 using the method described above.

Example 19.4: Given $R_{XX}(\tau) = e^{-\alpha|\tau|}$, we have

$$S_{XX}(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2} = \frac{N(\omega^2)}{D(\omega^2)}.$$

This gives $D^+(s) = \alpha + s,\ D^-(s) = \alpha - s$ and $P(s)$, $Q(s)$ are constants here. Moreover since $n = 1$, (19-102) reduces to $1 \pm a_1 = 0$, or $a_1 = \pm 1$ and from (19-101), ω_1 satisfies

$$e^{\omega_1 T} = \frac{D^-(\omega_1)}{D^+(\omega_1)} = \frac{\alpha - \omega_1}{\alpha + \omega_1} \qquad (19\text{-}113)$$

or ω_1 is the solution of the s-plane equation

$$e^{sT} = \frac{\alpha - s}{\alpha + s} \qquad (19\text{-}114)$$

But $|e^{sT}| > 1$ on the RHP, whereas $\left|\frac{\alpha - s}{\alpha + s}\right| < 1$ on the RHP. Similarly $|e^{sT}| < 1$ on the LHP, whereas $\left|\frac{\alpha - s}{\alpha + s}\right| > 1$ on the LHP.

Thus in (19-114) the solution s must be purely imaginary, and hence ω_1 in (19-113) is purely imaginary. Thus with $s = j\omega_1$ in (19-114) we get

$$e^{j\omega_1 T} = \frac{\alpha - j\omega_1}{\alpha + j\omega_1}$$

or

$$\tan(\omega_1 T/2) = -\frac{\omega_1}{\alpha} \qquad (19\text{-}115)$$

which agrees with the transcendental equation (19-65). Further from (19-108), the λs satisfy

$$D(-s^2) - \lambda_n N(-s^2)\Big|_{s=j\omega_n} = \alpha^2 + \omega_n^2 - 2\alpha\lambda_n = 0$$

or

$$\lambda_n = \frac{\alpha^2 + \omega_n^2}{2\alpha} > 0. \qquad (19\text{-}116)$$

Notice that the λ_n in (19-66) is the inverse of (19-116) because as noted earlier λ in (19-79) is the inverse of that in (19-22).

Finally from (19-112)

$$\varphi_n(t) = L^{-1}\left\{ \frac{s+\alpha}{s^2+\omega_n^2} \right\} = A_n \cos\omega_n t + B_n \sin\omega_n t, \qquad 0 < t < T \qquad (19\text{-}117)$$

which agrees with the solution obtained in (19-67). We conclude this section with a less trivial example.

Example 19.5

$$R_{XX}(\tau) = e^{-\alpha|\tau|} + e^{-\beta|\tau|}. \qquad (19\text{-}118)$$

In this case

$$S_{XX}(\omega) = \frac{2\alpha}{\omega^2+\alpha^2} + \frac{2\beta}{\omega^2+\beta^2} = \frac{2(\alpha+\beta)(\omega^2+\alpha\beta)}{(\omega^2+\alpha^2)(\omega^2+\beta^2)}. \qquad (19\text{-}119)$$

This gives $D^+(s) = (s+\alpha)(s+\beta) = s^2 + (\alpha+\beta)s + \alpha\beta$. With $n = 2$, (19-107) and its companion determinant reduce to

$$\omega_2 \tanh\, \theta_2 = \omega_1 \tanh\, \theta_1$$

$$\omega_2 \coth\, \theta_2 = \omega_1 \coth\, \theta_1$$

or

$$\tanh\,\theta_1 = \pm\tanh\,\theta_2. \tag{19-120}$$

From (19-106)

$$\tanh\,\theta_i = \frac{(\alpha\beta+\omega_i^2)\tanh\,(\omega_i T/2) + (\alpha+\beta)\omega_i}{(\alpha\beta+\omega_i^2) + (\alpha+\beta)\omega_i\tanh\,(\omega_i T/2)}, \qquad i=1,2 \tag{19-121}$$

Finally ω_1^2 and ω_2^2 can be parametrically expressed in terms of λ using (19-108) and it simplifies to

$$\begin{aligned} D(-s^2) - \lambda N(-s^2) &= s^4 - (\alpha^2+\beta^2 - 2\lambda(\alpha+\beta))s^2 \\ &\quad +\alpha^2\beta^2 - 2\lambda(\alpha+\beta)\alpha\beta \\ &\triangleq s^4 - bs^2 + c = 0. \end{aligned}$$

This gives

$$\omega_1^2 = \frac{b(\lambda) + \sqrt{b^2(\lambda) - 4c(\lambda)}}{2}$$

and

and

$$\omega_2^2 = \frac{b(\lambda) - \sqrt{b^2(\lambda) - 4c(\lambda)}}{2} = \omega_1^2 - \sqrt{b^2(\lambda) - 4c(\lambda)}$$

and substituting these into (19-120)-(19-121) the corresponding transcendental equation for $\lambda_i s$ can be obtained. Similarly the eigenfunctions can be obtained from (19-112).