

19. Series Representation of Stochastic Processes

Given information about a stochastic process $X(t)$ in $0 \leq t \leq T$, can this continuous information be represented in terms of a countable set of random variables whose relative importance decrease under some arrangement?

To appreciate this question it is best to start with the notion of a **Mean-Square periodic process**. A stochastic process $X(t)$ is said to be mean square (M.S) periodic, if for some $T > 0$

$$E[|X(t+T) - X(t)|^2] = 0 \quad \text{for all } t. \quad (19-1)$$

i.e $X(t) = X(t+T)$ with *probability* 1 for all t .

Suppose $X(t)$ is a W.S.S process. Then

$X(t)$ is mean-square periodic $\Leftrightarrow R(\tau)$ is periodic in the ordinary sense, where

$$R(\tau) = E[X(t)X^*(t+T)]$$

Proof: (\Rightarrow) suppose $X(t)$ is M.S. periodic. Then

$$E[|X(t+T) - X(t)|^2] = 0. \quad (19-2)$$

But from Schwarz' inequality

$$\left| E[X(t_1)\{X(t_2+T) - X(t_2)\}^*] \right|^2 \leq E[|X(t_1)|^2] \underbrace{E[|X(t_2+T) - X(t_2)|^2]}_0$$

Thus the left side equals

$$E[X(t_1)\{X(t_2+T) - X(t_2)\}^*] = 0$$

or

$$E[X(t_1)X^*(t_2+T)] = E[X(t_1)X^*(t_2)] \Rightarrow R(t_2 - t_1 + T) = R(t_2 - t_1)$$

$$\Rightarrow R(\tau + T) = R(\tau) \quad \text{for any } \tau$$

i.e., $R(\tau)$ is periodic with period T . (19-3)

(\Leftarrow) Suppose $R(\tau)$ is periodic. Then

$$E[|X(t+\tau) - X(t)|^2] = 2R(0) - R(\tau) - R^*(\tau) = 0$$

i.e., $X(t)$ is mean square periodic.

Thus if $X(t)$ is mean square periodic, then $R(\tau)$ is periodic and let

$$R(\tau) = \sum_{-\infty}^{+\infty} \gamma_n e^{jn\omega_0\tau}, \quad \omega_0 = \frac{2\pi}{T} \quad (19-4)$$

represent its Fourier series expansion. Here

$$\gamma_n = \frac{1}{T} \int_0^T R(\tau) e^{-jn\omega_0\tau} d\tau. \quad (19-5)$$

In a similar manner define

$$c_k = \frac{1}{T} \int_0^T X(t) e^{jk\omega_0 t} dt \quad (19-6)$$

Notice that $c_k, \quad k = -\infty \rightarrow +\infty$ are random variables, and

$$E[c_k c_m^*] = \frac{1}{T^2} E\left[\int_0^T X(t_1) e^{jk\omega_0 t_1} dt_1 \int_0^T X^*(t_2) e^{-jm\omega_0 t_2} dt_2 \right]$$

$$= \frac{1}{T^2} \int_0^T \int_0^T R(t_2 - t_1) e^{jk\omega_0 t_1} e^{-jm\omega_0 t_2} dt_1 dt_2$$

$$= \frac{1}{T} \int_0^T \underbrace{\left[\frac{1}{T} \int_0^T \underbrace{R(t_2 - t_1)}_{\tau} e^{-jm\omega_0 \overbrace{(t_2 - t_1)}^{\tau}} d \underbrace{(t_2 - t_1)}_{\tau} \right]}_{\gamma_m} e^{-j(m-k)\omega_0 t_1} dt_1$$

$$E[c_k c_m^*] = \gamma_m \underbrace{\left\{ \frac{1}{T} \int_0^T e^{-j(m-k)\omega_0 t_1} dt_1 \right\}}_{\delta_{m,k}} = \begin{cases} \gamma_m > 0, & k = m \\ 0 & k \neq m. \end{cases} \quad (19-7)$$

i.e., $\{c_n\}_{n=-\infty}^{n=+\infty}$ form a sequence of uncorrelated random variables, and, further, consider the partial sum

$$\tilde{X}_N(t) = \sum_{K=-N}^N c_k e^{-jk\omega_0 t}. \quad (19-8)$$

We shall show that $\tilde{X}_N(t) = X(t)$ in the mean square sense as $N \rightarrow \infty$. i.e.,

$$E[|X(t) - \tilde{X}_N(t)|^2] \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (19-9)$$

Proof:

$$\begin{aligned} E[|X(t) - \tilde{X}_N(t)|^2] &= E[|X(t)|^2] - 2 \operatorname{Re}[E(X^*(t) \tilde{X}_N(t))] \\ &\quad + E[|\tilde{X}_N(t)|^2]. \end{aligned} \quad (19-10)$$

But

$$E[|X(t)|^2] = R(0) = \sum_{k=-\infty}^{+\infty} \gamma_k,$$

and

$$\begin{aligned} E[X^*(t)\tilde{X}_N(t)] &= E\left[\sum_{k=-N}^N c_k e^{-jk\omega_0 t} X^*(t)\right] \\ &= \frac{1}{T} \sum_{k=-N}^N E\left[\int_0^T X(\alpha) e^{-jk\omega_0(t-\alpha)} X^*(t) d\alpha\right] \\ &= \sum_{k=-N}^N \underbrace{\left[\frac{1}{T} \int_0^T R(t-\alpha) e^{-jk\omega_0(t-\alpha)} d(t-\alpha)\right]}_{\gamma_k} = \sum_{k=-N}^N \gamma_k. \end{aligned} \quad (19-12)$$

Similarly

$$E[|\tilde{X}_N(t)|^2] = E\left[\sum_k \sum_m c_k c_m^* e^{j(k-m)\omega_0 t}\right] = \sum_k \sum_m E[c_k c_m^*] e^{j(k-m)\omega_0 t} = \sum_{k=-N}^N \gamma_k.$$

$$\Rightarrow E[|X(t) - \tilde{X}_N(t)|^2] = 2\left(\sum_{k=-\infty}^{+\infty} \gamma_k - \sum_{k=-N}^N \gamma_k\right) \rightarrow 0 \quad \text{as } N \rightarrow \infty \quad (19-13)$$

i.e.,

$$X(t) \doteq \sum_{k=-\infty}^{+\infty} c_k e^{-jk\omega_0 t}, \quad -\infty < t < +\infty. \quad (19-14)$$

Thus mean square periodic processes can be represented in the form of a series as in (19-14). The stochastic information is contained in the random variables c_k , $k = -\infty \rightarrow +\infty$. Further these random variables are uncorrelated ($E\{c_k c_m^*\} = \gamma_k \delta_{k,m}$) and their variances $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$. This follows by noticing that from (19-14)

$$\sum_{k=-\infty}^{+\infty} \gamma_k = R(0) = E[|X(t)|^2] = P < \infty.$$

Thus if the power P of the stochastic process is finite, then the positive sequence $\sum_{k=-\infty}^{+\infty} \gamma_k$ converges, and hence $\gamma_k \rightarrow 0$ as $k \rightarrow \infty$. This implies that the random variables in (19-14) are of relatively less importance as $k \rightarrow \infty$, and a finite approximation of the series in (19-14) is indeed meaningful.

The following natural question then arises: What about a general stochastic process, that is *not* mean square periodic? Can it be represented in a similar series fashion as in (19-14), if not in the whole interval $-\infty < t < \infty$, say in a finite support $0 \leq t \leq T$?

Suppose that it is indeed possible to do so for any arbitrary process $X(t)$ in terms of a certain sequence of orthonormal functions. PILLAI

i.e.,

$$\tilde{X}(t) = \sum_{n=1}^{\infty} c_n \varphi_n(t) \quad (19-15)$$

where

$$c_k \triangleq \int_0^T X(t) \varphi_k^*(t) dt \quad (19-16)$$

$$\int_0^T \varphi_k(t) \varphi_n^*(t) dt = \delta_{k,n}, \quad (19-17)$$

and in the mean square sense

$$\tilde{X}(t) \doteq X(t) \quad \text{in} \quad 0 \leq t \leq T.$$

Further, as before, we would like the c_k s to be uncorrelated random variables. If that should be the case, then we must have

$$E[c_k c_m^*] = \lambda_m \delta_{k,m}. \quad (19-18)$$

Now

$$\begin{aligned} E[c_k c_m^*] &= E\left[\int_0^T X(t_1) \varphi_k^*(t_1) dt_1 \int_0^T X^*(t_2) \varphi_m(t_2) dt_2\right] \\ &= \int_0^T \varphi_k^*(t_1) \int_0^T E\{X(t_1) X^*(t_2)\} \varphi_m(t_2) dt_2 dt_1 \\ &= \int_0^T \varphi_k^*(t_1) \left\{ \int_0^T R_{XX}(t_1, t_2) \varphi_m(t_2) dt_2 \right\} dt_1 \quad (19-19) \end{aligned}$$

and

$$\lambda_m \delta_{k,m} = \lambda_m \int_0^T \varphi_k^*(t_1) \varphi_m(t_1) dt_1. \quad (19-20)$$

Substituting (19-19) and (19-20) into (19-18), we get

$$\int_0^T \varphi_k^*(t_1) \left\{ \int_0^T R_{xx}(t_1, t_2) \varphi_m(t_2) dt_2 - \lambda_m \varphi_m(t_1) \right\} dt_1 = 0. \quad (19-21)$$

Since (19-21) should be true for *every* $\varphi_k(t)$, $k = 1 \rightarrow \infty$, we must have

$$\int_0^T R_{xx}(t_1, t_2) \varphi_m(t_2) dt_2 - \lambda_m \varphi_m(t_1) \equiv 0,$$

or

$$\int_0^T R_{xx}(t_1, t_2) \varphi_m(t_2) dt_2 = \lambda_m \varphi_m(t_1), \quad 0 < t_1 < T, \quad m = 1 \rightarrow \infty. \quad (19-22)$$

i.e., the desired uncorrelated condition in (19-18) gets translated into the integral equation in (19-22) and it is known as the *Karhunen-Loeve* or K-L. integral equation. The functions $\{\varphi_k(t)\}_{k=1}^{\infty}$ are *not arbitrary* and they must be obtained by solving the integral equation in (19-22). They are known as the eigenvectors of the autocorrelation

function of $R_{xx}(t_1, t_2)$. Similarly the set $\{\lambda_k\}_{k=1}^{\infty}$ represent the eigenvalues of the autocorrelation function. From (19-18), the eigenvalues λ_k represent the variances of the uncorrelated random variables c_k , $k = 1 \rightarrow \infty$. This also follows from Mercer's theorem which allows the representation

$$R_{xx}(t_1, t_2) = \sum_{k=1}^{\infty} \mu_k \phi_k(t_1) \phi_k^*(t_2), \quad 0 < t_1, t_2 < T, \quad (19-23)$$

where

$$\int_0^T \phi_k(t) \phi_m^*(t) dt = \delta_{k,m}.$$

Here $\phi_k(t)$ and μ_k , $k = 1 \rightarrow \infty$ are known as the eigenfunctions and eigenvalues of $R_{xx}(t_1, t_2)$ respectively. A direct substitution and simplification of (19-23) into (19-22) shows that

$$\varphi_k(t) = \phi_k(t), \quad \lambda_k = \mu_k, \quad k = 1 \rightarrow \infty. \quad (19-24)$$

Returning back to (19-15), once again the partial sum

$$\tilde{X}_N(t) = \sum_{k=1}^N c_k \varphi_k(t) \xrightarrow{N \rightarrow \infty} X(t), \quad 0 \leq t \leq T \quad (19-25)$$

in the mean square sense. To see this, consider

$$\begin{aligned} E[|X(t) - \tilde{X}_N(t)|^2] &= E[|X(t)|^2] - E[X(t)\tilde{X}_N^*(t)] \\ &\quad - E[X^*(t)\tilde{X}_N(t)] + E[|\tilde{X}_N(t)|^2]. \end{aligned} \quad (19-26)$$

We have

$$E[|X(t)|^2] = R(t, t). \quad (19-27)$$

Also

$$\begin{aligned} E[X(t)\tilde{X}_N^*(t)] &= \sum_{k=1}^N X(t)c_k^* \phi_k^*(t) \\ &= \sum_{k=1}^N \int_0^T E[X(t)X^*(\alpha)] \phi_k^*(t)\phi_k(\alpha) d\alpha \\ &= \sum_{k=1}^N \left(\int_0^T R(t, \alpha)\phi_k(\alpha) d\alpha \right) \phi_k^*(t) \\ &= \sum_{k=1}^N \lambda_k \phi_k(t)\phi_k^*(t) = \sum_{k=1}^N \lambda_k |\phi_k(t)|^2. \end{aligned} \quad (19-28)$$

Similarly

$$E[X^*(t)\tilde{X}_N(t)] = \sum_{k=1}^N \lambda_k |\phi_k(t)|^2 \quad (19-29)$$

and

$$E[|\tilde{X}_N(t)|^2] = \sum_k \sum_m E[c_k c_m^*] \varphi_k(t) \varphi_m^*(t) = \sum_{k=1}^N \lambda_k |\varphi_k(t)|^2. \quad (19-30)$$

Hence (19-26) simplifies into

$$E[|X(t) - \tilde{X}_N(t)|^2] = R(t, t) - \sum_{k=1}^N \lambda_k |\varphi_k(t)|^2 \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (19-31)$$

i.e.,

$$X(t) \doteq \sum_{k=1}^{\infty} c_k \varphi_k(t), \quad 0 \leq t \leq T, \quad (19-32)$$

where the random variables $\{c_k\}_{k=1}^{\infty}$ are uncorrelated and faithfully represent the random process $X(t)$ in $0 \leq t \leq T$, provided $\varphi_k(t)$, $k = 1 \rightarrow \infty$, satisfy the K-L. integral equation.

Example 19.1: If $X(t)$ is a w.s.s white noise process, determine the sets $\{\varphi_k, \lambda_k\}_{k=1}^{\infty}$ in (19-22).

Solution: Here

$$R_{XX}(t_1, t_2) = q\delta(t_1 - t_2) \quad (19-33)$$

and

$$\begin{aligned} \int_0^T R_{xx}(t_1, t_2) \varphi_k(t_2) dt_1 &= q \int_0^T \delta(t_1 - t_2) \varphi_k(t_2) dt_1 \\ &= q \varphi_k(t_1) \stackrel{\Delta}{=} \lambda_k \varphi_k(t_1) \end{aligned} \quad (19-34)$$

$\Rightarrow \varphi_k(t)$ can be arbitrary so long as they are orthonormal as in (19-17) and $\lambda_k = q, \quad k = 1 \rightarrow \infty$. Then the power of the process

$$P = E[|X(t)|^2] = R(0) = \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} q = \infty$$

and in that sense white noise processes are *unrealizable*. However, if the received waveform is given by

$$r(t) = s(t) + n(t), \quad 0 < t < T \quad (19-35)$$

and $n(t)$ is a w.s.s white noise process, then since *any* set of orthonormal functions is sufficient for the white noise process representation, they can be chosen solely by considering the other signal $s(t)$. Thus, in (19-35)

$$R_{rr}(t_1 - t_2) = R_{ss}(t_1 - t_2) + q\delta(t_1 - t_2) \quad (19-36)$$

and if

$$R_{ss}(t_1 - t_2) = \sum_{k=1}^{\infty} \lambda_k \phi_k(t_1) \phi_k^*(t_2) \quad (19-37)$$

Then it follows that

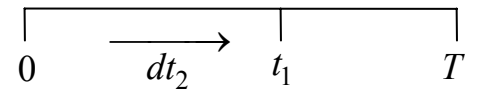
$$R_{rr}(t_1 - t_2) = \sum_{k=1}^{\infty} (\lambda_k + q) \phi_k(t_1) \phi_k^*(t_2). \quad (19-38)$$

Notice that the eigenvalues of $R_{ss}(t_1 - t_2)$ get incremented by q .

Example 19.2: $X(t)$ is a Wiener process with

$$R_{xx}(t_1, t_2) = \alpha \min(t_1, t_2) = \begin{cases} \alpha t_2 & t_1 > t_2 \\ \alpha t_1 & t_1 \leq t_2 \end{cases}, \quad \alpha > 0 \quad (19-39)$$

In that case Eq. (19-22) simplifies to



$$\begin{aligned} \int_0^T R_{XX}(t_1, t_2) \phi_k(t_2) dt_2 &= \int_0^{t_1} R_{XX}(t_1, t_2) \phi_k(t_2) dt_2 \\ &+ \int_{t_1}^T R_{XX}(t_1, t_2) \phi_k(t_2) dt_2 = \lambda_k \phi_k(t_1), \end{aligned}$$

and using (19-39) this simplifies to

$$\int_0^{t_1} \alpha t_2 \phi_k(t_2) dt_2 + \int_{t_1}^T \alpha t_1 \phi_k(t_2) dt_2 = \lambda_k \phi_k(t_1). \quad (19-40)$$

Derivative with respect to t_1 gives [see Eqs. (8-5)-(8-6), Lecture 8]

$$\alpha t_1 \varphi_k(t_1) + (-1)\alpha t_1 \dot{\varphi}_k(t_1) + \alpha \int_{t_1}^T \varphi_k(t_2) dt_2 = \lambda_k \dot{\varphi}_k(t_1)$$

or

$$\alpha \int_{t_1}^T \varphi_k(t_2) dt_2 = \lambda_k \dot{\varphi}_k(t_1). \quad (19-41)$$

Once again, taking derivative with respect to t_1 , we obtain

$$\alpha(-1)\varphi_k(t_1) = \lambda_k \ddot{\varphi}_k(t_1)$$

or

$$\frac{d^2 \varphi_k(t_1)}{dt_1^2} + \frac{\alpha}{\lambda_k} \varphi_k(t_1) = 0, \quad (19-42)$$

and its solution is given by

$$\varphi_k(t) = A_k \cos \sqrt{\frac{\alpha}{\lambda_k}} t + B_k \sin \sqrt{\frac{\alpha}{\lambda_k}} t.$$

But from (19-40)

$$\varphi_k(0) = 0, \quad (19-43)$$

and from (19-41)

$$\dot{\phi}_k(T) = 0. \quad (19-44)$$

This gives

$$\begin{aligned} \phi_k(0) = A_k = 0, \quad k = 1 \rightarrow \infty, \\ \dot{\phi}_k(t) = B_k \sqrt{\frac{\alpha}{\lambda_k}} \cos \sqrt{\frac{\alpha}{\lambda_k}} t, \end{aligned} \quad (19-45)$$

and using (19-44) we obtain

$$\begin{aligned} \dot{\phi}_k(T) = B_k \sqrt{\frac{\alpha}{\lambda_k}} \cos \sqrt{\frac{\alpha}{\lambda_k}} T = 0 \\ \Rightarrow \sqrt{\frac{\alpha}{\lambda_k}} T = (2k - 1) \frac{\pi}{2} \end{aligned} \quad (19-46)$$

$$\Rightarrow \lambda_k = \frac{\alpha T^2}{(k - \frac{1}{2})^2 \pi^2}, \quad k = 1 \rightarrow \infty. \quad (19-47)$$

Also

$$\varphi_k(t) = B_k \sin \sqrt{\frac{\alpha}{\lambda_k}} t, \quad 0 \leq t \leq T. \quad (19-48)$$

Further, orthonormalization gives

$$\begin{aligned} \int_0^T \varphi_k^2(t) dt &= B_k^2 \int_0^T \left(\sin \sqrt{\frac{\alpha}{\lambda_k}} t \right)^2 dt = B_k^2 \left[\int_0^T \left(\frac{1 - \cos 2\sqrt{\frac{\alpha}{\lambda_k}} t}{2} \right) dt \right] \\ &= B_k^2 \left(\frac{T}{2} - \frac{1}{2} \frac{\sin 2\sqrt{\frac{\alpha}{\lambda_k}} t}{2\sqrt{\frac{\alpha}{\lambda_k}}} \Big|_0^T \right) = B_k^2 \left(\frac{T}{2} - \frac{\sin(2k-1)\pi - 0}{4\sqrt{\frac{\alpha}{\lambda_k}}} \right) = B_k^2 \frac{T}{2} = 1 \\ \Rightarrow B_k &= \sqrt{2/T}. \end{aligned}$$

Hence

$$\varphi_k(t) = \sqrt{\frac{2}{T}} \sin \left(\sqrt{\frac{\alpha}{\lambda_k}} t \right) = \sqrt{\frac{2}{T}} \sin \left(k - \frac{1}{2} \right) \frac{\pi t}{T}, \quad (19-49)$$

with λ_k as in (19-47) and c_k as in (19-16),

$X(t) = \sum_{k=1}^{\infty} c_k \varphi_k(t)$ is the desired series representation.

Example 19.3: Given

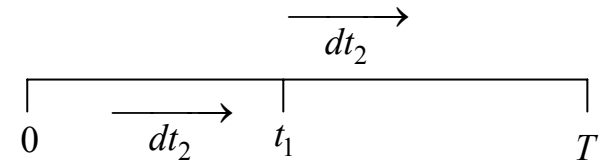
$$R_{XX}(\tau) = e^{-\alpha|\tau|}, \quad \alpha > 0, \quad (19-50)$$

find the orthonormal functions for the series representation of the underlying stochastic process $X(t)$ in $0 < t < T$.

Solution: We need to solve the equation

$$\int_0^T e^{-\alpha|t_1-t_2|} \varphi_n(t_2) dt_2 = \lambda_n \varphi_n(t_1). \quad (19-51)$$

Notice that (19-51) can be rewritten as,



$$\int_0^{t_1} e^{-\alpha \overbrace{(t_1-t_2)}^{\geq 0}} \varphi_n(t_2) dt_2 + \int_{t_1}^T e^{-\alpha \overbrace{(t_2-t_1)}^{\geq 0}} \varphi_n(t_2) dt_2 = \lambda_n \varphi_n(t_1) \quad (19-52)$$

Differentiating (19-52) once with respect to t_1 , we obtain

$$\begin{aligned}
& \varphi_n(t_1) + \int_0^{t_1} (-\alpha) e^{-\alpha(t_1-t_2)} \varphi_n(t_2) dt_2 - \varphi_n(t_1) + \int_{t_1}^T \alpha e^{-\alpha(t_2-t_1)} \varphi_n(t_2) dt_2 \\
&= \lambda_n \frac{d\varphi_n(t_1)}{dt_1} \\
\Rightarrow & -\int_0^{t_1} e^{-\alpha(t_1-t_2)} \varphi_n(t_2) dt_2 + \int_{t_1}^T e^{-\alpha(t_2-t_1)} \varphi_n(t_2) dt_2 = \frac{\lambda_n}{\alpha} \frac{d\varphi_n(t_1)}{dt_1} \quad (19-53)
\end{aligned}$$

Differentiating (19-53) again with respect to t_1 , we get

$$\begin{aligned}
& -\varphi_n(t_1) - \int_0^{t_1} (-\alpha) e^{-\alpha(t_1-t_2)} \varphi_n(t_2) dt_2 \\
& -\varphi_n(t_1) + \int_{t_1}^T \alpha e^{-\alpha(t_2-t_1)} \varphi_n(t_2) dt_2 = \frac{\lambda_n}{\alpha} \frac{d^2\varphi_n(t_1)}{dt_1^2}
\end{aligned}$$

or

$$\begin{aligned}
& -2\varphi_n(t_1) + \alpha \underbrace{\left[\int_0^{t_1} e^{-\alpha(t_1-t_2)} \varphi_n(t_2) dt_2 + \int_{t_1}^T e^{-\alpha(t_2-t_1)} \varphi_n(t_2) dt_2 \right]}_{\lambda_n \varphi_n(t_1) \text{ \{use (19-52)\}}} \\
&= \frac{\lambda_n}{\alpha} \frac{d^2\varphi_n(t_1)}{dt_1^2}
\end{aligned}$$

or

$$(\alpha\lambda_n - 2)\varphi_n(t_1) = \frac{\lambda_n}{\alpha} \frac{d^2\varphi_n(t_1)}{dt_1^2}$$

or

$$\frac{d^2\varphi_n(t_1)}{dt_1^2} = \left(\frac{\alpha(\alpha\lambda_n - 2)}{\lambda_n} \right) \varphi_n(t_1). \quad (19-54)$$

Eq.(19-54) represents a second order differential equation. The solution for $\varphi_n(t)$ depends on the value of the constant $\alpha(\alpha\lambda_n - 2)/\lambda_n$ on the right side. We shall show that solutions exist in this case only if

$$\alpha\lambda_n < 2, \text{ or} \quad 0 < \lambda_n < \frac{2}{\alpha}. \quad (19-55)$$

In that case $\alpha(\alpha\lambda_n - 2)/\lambda_n < 0$.

Let

$$\omega_n^2 \triangleq \frac{\alpha(2 - \alpha\lambda_n)}{\lambda_n} > 0, \quad (19-56)$$

and (19-54) simplifies to

$$\frac{d^2\varphi_n(t_1)}{dt_1^2} = -\omega_n^2 \varphi_n(t_1). \quad (19-57)$$

General solution of (19-57) is given by

$$\varphi_n(t_1) = A_n \cos \omega_n t_1 + B_n \sin \omega_n t_1. \quad (19-58)$$

From (19-52)

$$\varphi_n(0) = \frac{1}{\lambda_n} \int_0^T e^{-\alpha t_2} \varphi_n(t_2) dt_2 \quad (19-59)$$

and

$$\varphi_n(T) = \frac{1}{\lambda_n} \int_0^T e^{-\alpha t_2} \varphi_n(T - t_2) dt_2. \quad (19-60)$$

Similarly from (19-53)

$$\dot{\varphi}_n(0) = \left. \frac{d\varphi_n(t_1)}{dt_1} \right|_{t_1=0} = \frac{\alpha}{\lambda_n} \int_0^T e^{-\alpha t_2} \varphi_n(t_2) dt_2 = \alpha \varphi_n(0) \quad (19-61)$$

and

$$\dot{\varphi}_n(T) = -\frac{\alpha}{\lambda_n} \int_0^T e^{-\alpha t_2} \varphi_n(T - t_2) dt_2 = -\alpha \varphi_n(T). \quad (19-62)$$

Using (19-58) in (19-61) gives

$$B_n \omega_n = \alpha A_n$$

or

$$\frac{A_n}{B_n} = \frac{\omega_n}{\alpha}, \quad (19-63)$$

and using (19-58) in (19-62), we have

$$\begin{aligned} -A_n \omega_n \sin \omega_n T + B_n \omega_n \cos \omega_n T &= -\alpha (A_n \cos \omega_n T + B_n \sin \omega_n T), \\ \Rightarrow (A_n \alpha + B_n \omega_n) \cos \omega_n T &= (A_n \omega_n - B_n \alpha) \sin \omega_n T \end{aligned}$$

or

$$\tan \omega_n T = \frac{2A_n \alpha}{A_n \omega_n - B_n \alpha} = \frac{2A_n \alpha / B_n \alpha}{\frac{A_n \omega_n}{B_n \alpha} - 1} = \frac{2A_n / B_n}{\frac{A_n \omega_n}{B_n \alpha} - 1} = \frac{2(\omega_n / \alpha)}{\left(\frac{\omega_n}{\alpha}\right)^2 - 1}.$$

Thus ω_n s are obtained as the solution of the transcendental equation

$$\tan \omega_n T = \frac{2(\omega_n / \alpha)}{(\omega_n / \alpha)^2 - 1}, \quad (19-64)$$

which simplifies to

$$\tan(\omega_n T / 2) = -\frac{\omega_n}{\alpha}. \quad (19-65)$$

In terms of ω_n s from (19-56) we get

$$\lambda_n = \frac{2\alpha}{\alpha^2 + \omega_n^2} > 0. \quad (19-66)$$

Thus the eigenvalues are obtained as the solution of the transcendental equation (19-65). (see Fig 19.1). For each such λ_n (or ω_n^2), the corresponding eigenvector is given by (19-58). Thus

$$\begin{aligned} \varphi_n(t) &= A_n \cos \omega_n t + B_n \sin \omega_n t \\ &= c_n \sin(\omega_n t - \theta_n) = c_n \sin \omega_n \left(t - \frac{T}{2}\right), \quad 0 < t < T \end{aligned} \quad (19-67)$$

since from (19-65)

$$\theta_n = \tan^{-1} \left(-\frac{A_n}{B_n} \right) = \tan^{-1} \left(-\frac{\omega_n}{\alpha} \right) = \omega_n T / 2, \quad (19-68)$$

and c_n is a suitable normalization constant.

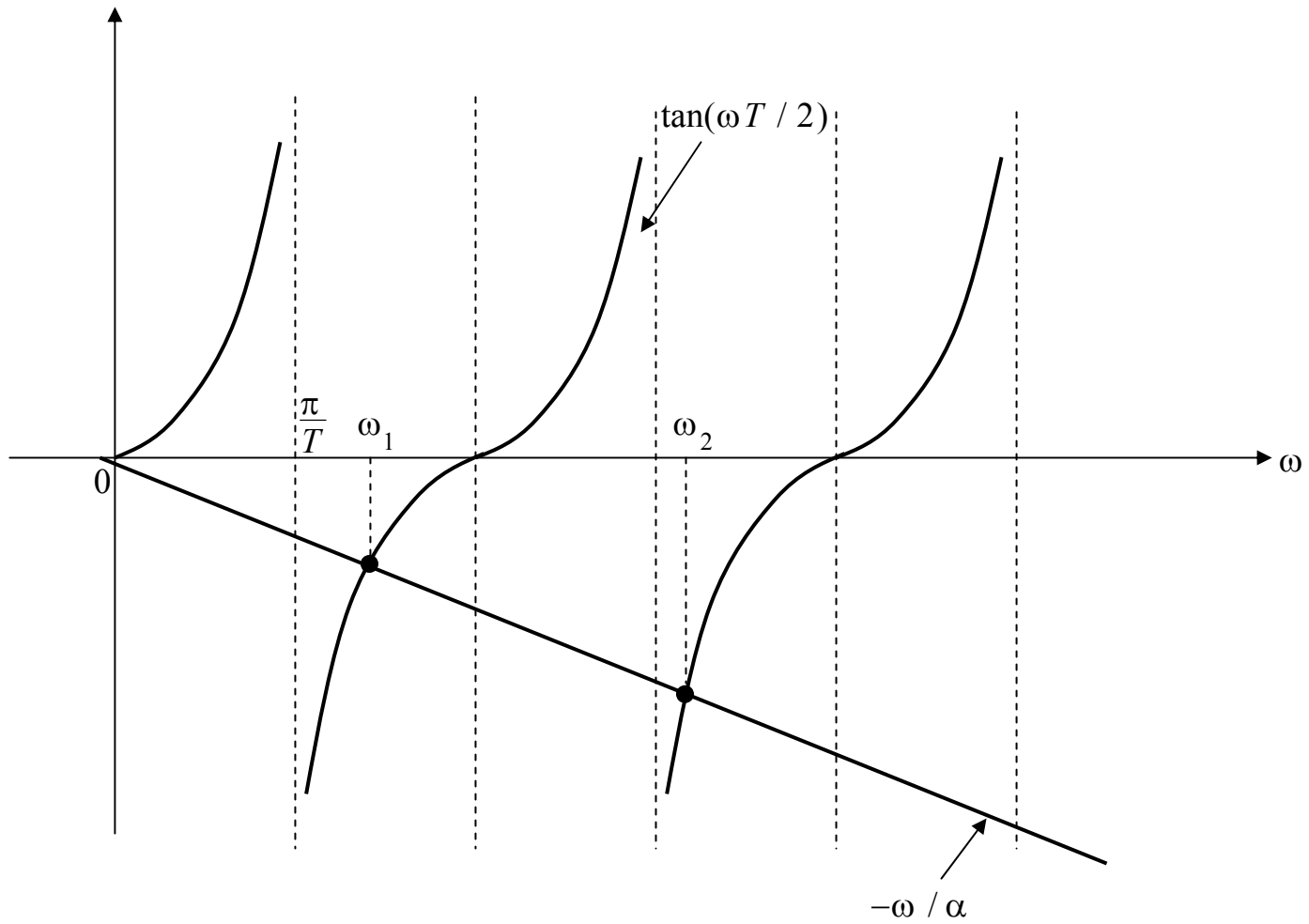


Fig 19.1 Solution for Eq.(19-65).

Karhunen – Loeve Expansion for Rational Spectra

[The following exposition is based on Youla's classic paper "The solution of a Homogeneous Wiener-Hopf Integral Equation occurring in the expansion of Second-order Stationary Random Functions," IRE Trans. on Information Theory, vol. 9, 1957, pp 187-193. Youla is tough. Here is a friendlier version. Even this may be skipped on a first reading. (Youla can be made only so much friendly.)]

Let $X(t)$ represent a w.s.s zero mean real stochastic process with autocorrelation function $R_{xx}(\tau) = R_{xx}(-\tau)$ so that its power spectrum

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = 2 \int_0^{\infty} R_{xx}(\tau) \cos\tau d\tau \quad (19-69)$$

is nonnegative and an even function. If $S_{xx}(\omega)$ is rational, then the process $X(t)$ is said to be rational as well. $S_{xx}(\omega)$ rational and even implies

$$S_{xx}(\omega) = \frac{N(\omega^2)}{D(\omega^2)} \geq 0. \quad (19-70)$$

The total power of the process is given by

$$P = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{xx}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{N(\omega^2)}{D(\omega^2)} d\omega \quad (19-71)$$

and for P to be finite, we must have

(i) The degree $\delta(D) = 2n$ of the denominator polynomial $D(\omega^2)$ must exceed the degree $\delta(N) = 2m$ of the numerator polynomial $N(\omega^2)$ by at least two,

and

(ii) $D(\omega^2)$ must not have any zeros on the real-frequency ($s = j\omega$) axis.

The s -plane ($s = \sigma + j\omega$) extension of $S_{xx}(\omega)$ is given by

$$S_{xx}(\omega) \Big|_{s=j\omega} \triangleq S(s^2) = \frac{N(-s^2)}{D(-s^2)}. \quad (19-72)$$

Thus

$$D(-s^2) = \prod_k (s^2 - \mu_k^2)^{k_i} \quad (19-73)$$

and the Laplace inverse transform of $(s^2 - \alpha^2)^{-k}$ is given by 25
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$$\frac{1}{(s^2 - \alpha^2)^k} \leftrightarrow \frac{(-1)^k}{(k-1)!} e^{-\alpha|\tau|} \sum_{j=1}^k \frac{(k+j-2)!}{(j-1)!(k-j)!} \frac{|\tau|^{k-j}}{(2\alpha)^{k+j-1}} \quad (19-74)$$

Let $\pm\mu_1, \pm\mu_2, \dots, \pm\mu_n$ represent the roots of $D(-s^2)$. Then

$$0 < \text{Re } \mu_1 \leq \text{Re } \mu_2 \leq \dots \leq \text{Re } \mu_n \quad (19-75)$$

Let $D^+(s)$ and $D^-(s)$ represent the left half plane (LHP) and the right half plane (RHP) products of these roots respectively. Thus

$$D(-s^2) = D^+(s)D^-(s), \quad (19-76)$$

where

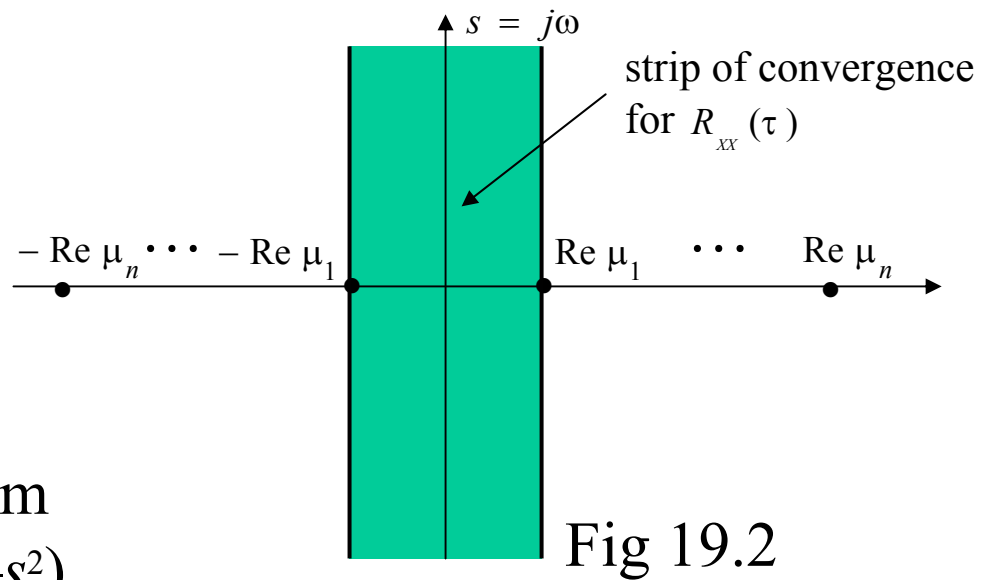
$$D^+(s) = \prod_k (s + \mu_k)(s + \mu_k^*) = \sum_{k=0}^n d_k s^k = D^-(-s). \quad (19-77)$$

This gives

$$S(s^2) = \frac{N(-s^2)}{D(-s^2)} = \frac{C_1(s)}{D^+(s)} + \frac{C_2(s)}{D^-(s)} \quad (19-78)$$

Notice that $\frac{C_1(s)}{D^+(s)}$ has poles only on the LHP and its inverse (for all $t > 0$) converges only if the strip of convergence is to the right

of *all* its poles. Similarly $C_2(s) / D^-(s)$ has poles only on the RHP and its inverse will converge only if the strip is to the left of all those poles. In that case, the inverse exists for $t < 0$. In the case of $R_{xx}(\tau)$, from (19-78) its transform $N(s^2) / D(-s^2)$



is defined only for $-\text{Re } \mu_1 < \text{Re } s < \text{Re } \mu_1$ (see Fig 19.2). In particular, for $\tau > 0$, from the above discussion it follows that $R_{xx}(\tau)$ is given by the inverse transform of $C_1(s) / D^+(s)$. We need the solution to the integral equation

$$\varphi(t) = \lambda \int_0^T R_{xx}(t-\tau)\varphi(\tau)d\tau, \quad 0 < t < T \quad (19-79)$$

that is valid *only* for $0 < t < T$. (Notice that λ in (19-79) is the reciprocal of the eigenvalues in (19-22)). On the other hand, the right side (19-79) can be defined for every t . Thus, let

$$g(t) \triangleq \int_0^T R_{xx}(t-\tau)\varphi(\tau)d\tau, \quad -\infty < t < +\infty \quad (19-80)$$

and to confirm with the same limits, define

$$\phi(t) = \begin{cases} \varphi(t) & 0 < t < T \\ 0 & \text{otherwise} \end{cases}. \quad (19-81)$$

This gives

$$g(t) = \int_{-\infty}^{+\infty} R_{xx}(t-\tau)\phi(\tau)d\tau \quad (19-82)$$

and let

$$f(t) = \phi(t) - \lambda g(t) = \phi(t) - \lambda \int_{-\infty}^{+\infty} R_{xx}(t-\tau)\phi(\tau)d\tau. \quad (19-83)$$

Clearly

$$f(t) = 0, \quad 0 < t < T \quad (19-84)$$

and for $t > T$

$$D^+ \left(\frac{d}{dt} \right) f(t) = -\lambda \int_{-\infty}^{+\infty} \overbrace{\left\{ D^+ \left(\frac{d}{dt} \right) R_{xx}(t-\tau) \right\}}^{D^+(-\mu_k)=0} \phi(\tau)d\tau = 0, \quad (19-85)$$

since $R_{xx}(t)$ is a sum of exponentials $\sum_k a_k e^{-\mu_k t}$, for $t > 0$. Hence it follows that for $t > T$, the function $f(t)$ must be a sum of exponentials $\sum_k a_k e^{-\mu_k t}$. Similarly for $t < 0$

$$D^- \left(\frac{d}{dt} \right) f(t) = -\lambda \int_{-\infty}^{+\infty} \overbrace{\left\{ D^- \left(\frac{d}{dt} \right) R_{xx}(t-\tau) \right\}}^{D^-(\mu_k)=0} \phi(\tau) d\tau = 0,$$
 and hence $f(t)$ must be a sum of exponentials $\sum_k b_k e^{\mu_k t}$, for $t < 0$. Thus the overall Laplace transform of $f(t)$ has the form

$$F(s) = \underbrace{\frac{P(s)}{D^-(s)}}_{\substack{\text{contributions} \\ \text{in } t < 0}} - e^{-sT} \underbrace{\frac{Q(s)}{D^+(s)}}_{\substack{\text{contributes to } t > 0 \\ \text{contributions in } t > T}} \quad (19-86)$$

where $P(s)$ and $Q(s)$ are polynomials of degree $n - 1$ at most. Also from (19-83), the bilateral Laplace transform of $f(t)$ is given by

$$F(s) = \Phi(s) \left[1 - \lambda \frac{N(-s^2)}{D(-s^2)} \right], \quad -\text{Re } \mu_1 < \text{Re } s < \text{Re } \mu_1 \quad (19-87)$$

Equating (19-86) and (19-87) and simplifying, Youla obtains the key identity

$$\Phi(s) = \frac{P(s)D^+(s) - e^{-sT}Q(s)D^-(s)}{D(-s^2) - \lambda N(-s^2)}. \quad (19-88)$$

Youla argues as follows: The function $\Phi(s) = \int_0^T \phi(t)e^{-st} dt$ is an entire function of s , and hence it is free of *poles* on the *entire*

finite s -plane ($-\infty < \operatorname{Re} s < +\infty$). However, the denominator on the right side of (19-88) is a polynomial and its roots contribute to poles of $\Phi(s)$. Hence all such poles must be cancelled by the numerator. As a result the numerator of $\Phi(s)$ in (19-88) must possess exactly the *same set* of zeros as its denominator to the respective order at least.

Let $\pm\omega_1(\lambda), \pm\omega_2(\lambda), \dots, \pm\omega_n(\lambda)$ be the (distinct) zeros of the denominator polynomial $D(-s^2) - \lambda N(-s^2)$. Here we assume that λ is an eigenvalue for which all ω_k 's are distinct. We have

$$0 < \operatorname{Re}\omega_1(\lambda) < \operatorname{Re}\omega_2(\lambda) < \dots < \operatorname{Re}\omega_n(\lambda) < \infty. \quad (19-89)$$

These ω_k 's also represent the zeros of the numerator polynomial $P(s)D^+(s) - e^{-sT}Q(s)D^-(s)$. Hence

$$D^+(\omega_k)P(\omega_k) = e^{-\omega_k T} D^-(\omega_k)Q(\omega_k) \quad (19-90)$$

and

$$D^+(-\omega_k)P(-\omega_k) = e^{\omega_k T} D^-(-\omega_k)Q(-\omega_k) \quad (19-91)$$

which simplifies into

$$D^-(\omega_k)P(-\omega_k) = e^{\omega_k T} D^+(\omega_k)Q(-\omega_k). \quad (19-92)$$

From (19-90) and (19-92) we get

$$P(\omega_k)P(-\omega_k) = Q(\omega_k)Q(-\omega_k), \quad k = 1, 2, \dots, n \quad (19-93)$$

i.e., the polynomial

$$L(s) = P(s)P(-s) - Q(s)Q(-s) \quad (19-94)$$

which is at most of degree $n - 1$ in s^2 vanishes at $\omega_1^2, \omega_2^2, \dots, \omega_n^2$ (for n distinct values of s^2). Hence

$$L(s^2) \equiv 0 \quad (19-95)$$

or

$$P(s)P(-s) = Q(s)Q(-s). \quad (19-96)$$

Using the linear relationship among the coefficients of $P(s)$ and $Q(s)$ in (19-90)-(19-91) it follows that

$$P(s) = \pm Q(s) \quad \text{or} \quad P(s) = \pm Q(-s) \quad (19-97)$$

are the only solutions that are consistent with each of those equations, and together we obtain

$$P(s) = \pm Q(-s) \quad (19-98)$$

as the *only solution* satisfying both (19-90) and (19-91). Let

$$P(s) = \sum_{i=0}^{n-1} p_i s^i. \quad (19-99)$$

In that case (19-90)-(19-91) simplify to (use (19-98))

$$\begin{aligned} P(\omega_k) D^+(\omega_k) \mp e^{-\omega_k T} D^-(\omega_k) P(-\omega_k) \\ = \sum_{i=0}^{n-1} \{1 \mp (-1)^i a_k\} \omega_k^i p_i = 0, \quad k = 1, 2, \dots, n \end{aligned} \quad (19-100)$$

where

$$a_k = \frac{D^-(\omega_k)}{D^+(\omega_k)} e^{-\omega_k T} = \frac{D^+(-\omega_k)}{D^+(\omega_k)} e^{-\omega_k T}. \quad (19-101)$$

For a nontrivial solution to exist for p_0, p_1, \dots, p_{n-1} in (19-100), we must have

$$\Delta_{1,2} = \begin{vmatrix} (1 \mp a_1) & (1 \pm a_1)\omega_1 & \cdots & (1 \mp (-1)^{n-1} a_1)\omega_1^{n-1} \\ (1 \mp a_2) & (1 \pm a_2)\omega_2 & \cdots & (1 \mp (-1)^{n-1} a_2)\omega_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ (1 \mp a_n) & (1 \pm a_n)\omega_n & \cdots & (1 \mp (-1)^{n-1} a_n)\omega_n^{n-1} \end{vmatrix} = 0. \quad (19-102)$$

The two determinant conditions in (19-102) must be solved together to obtain the eigenvalues λ_i 's that are implicitly contained in the a_i 's and ω_i 's (Easily said than done!).

To further simplify (19-102), one may express a_k in (19-101) as

$$a_k = e^{-2\theta_k}, \quad k = 1, 2, \dots, n \quad (19-103)$$

so that

$$\begin{aligned} \tanh \theta_k &= \frac{e^{\theta_k} - e^{-\theta_k}}{e^{\theta_k} + e^{-\theta_k}} = \frac{1 - a_k}{1 + a_k} = \frac{D^+(\omega_k) - e^{-\omega_k T} D^+(-\omega_k)}{D^+(\omega_k) + e^{-\omega_k T} D^+(-\omega_k)} \\ &= \frac{e^{\omega_k T/2} D^+(\omega_k) - e^{-\omega_k T/2} D^+(-\omega_k)}{e^{\omega_k T/2} D^+(\omega_k) + e^{-\omega_k T/2} D^+(-\omega_k)} \end{aligned} \quad (19-104)$$

Let

$$D^+(s) = d_0 + d_1s + \cdots + d_n s^n \quad (19-105)$$

and substituting these known coefficients into (19-104) and simplifying we get

$$\tanh \theta_k = \frac{(d_0 + d_2 \omega_k^2 + \cdots) \tanh (\omega_k T / 2) + (d_1 \omega_k + d_3 \omega_k^3 + \cdots)}{(d_0 + d_2 \omega_k^2 + \cdots) + (d_1 \omega_k + d_3 \omega_k^3 + \cdots) \tanh (\omega_k T / 2)} \quad (19-106)$$

and in terms of $\tanh \theta_k$, Δ_2 in (19-102) simplifies to

$$\begin{vmatrix} 1 & \omega_1 \tanh \theta_1 & \omega_1^2 & \omega_1^3 \tanh \theta_1 & \cdots & \omega_1^{n-1} \tanh \theta_1 \\ 1 & \omega_2 \tanh \theta_2 & \omega_2^2 & \omega_2^3 \tanh \theta_2 & \cdots & \omega_2^{n-1} \tanh \theta_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega_n \tanh \theta_n & \omega_n^2 & \omega_n^3 \tanh \theta_n & \cdots & \omega_n^{n-1} \tanh \theta_n \end{vmatrix} = 0 \quad (19-107)$$

if n is even (if n is odd the last column in (19-107) is simply

$[\omega_1^{n-1}, \omega_2^{n-1}, \cdots, \omega_n^{n-1}]^T$). Similarly Δ_1 in (19-102) can be obtained by replacing $\tanh \theta_k$ with $\coth \theta_k$ in (19-107).

To summarize determine the roots ω_k 's with $\text{Re}(\omega_i) > 0$ that satisfy

$$D(-\omega_k^2) - \lambda N(-\omega_k^2) = 0, \quad k = 1, 2, \dots, n \quad (19-108)$$

in terms of λ , and for every such ω_k , determine θ_k using (19-106). Finally using these ω_k 's and $\tanh \theta_k$'s in (19-107) and its companion equation Δ_1 , the eigenvalues λ_k 's are determined. Once λ_k 's are obtained, p_k 's can be solved using (19-100), and using that $\Phi_i(s)$ can be obtained from (19-88).

Thus

$$\Phi_i(s) = \frac{D^+(s)P(s, \lambda_i) - e^{-sT} D^-(s)Q(s, \lambda_i)}{D(-s^2) - \lambda_i N(-s^2)} \quad (19-109)$$

and

$$\phi_i(t) = L^{-1}\{\Phi_i(s)\}. \quad (19-110)$$

Since $\Phi_i(s)$ is an entire function in (19-110), the inverse Laplace transform in (19-109) can be performed through *any* strip of convergence in the s-plane, and in particular if we use the strip

$\text{Re } s > \text{Re}(\omega_n)$ (to the right of all $\text{Re}(\omega_i)$), then the two inverses

$$L^{-1} \left\{ \frac{D^+(s)P(s)}{D(-s^2) - \lambda N(-s^2)} \right\}, \quad L^{-1} \left\{ \frac{D^-(s)Q(s)}{D(-s^2) - \lambda N(-s^2)} \right\} \quad (19-111)$$

obtained from (19-109) will be causal. As a result $L^{-1} \left\{ e^{-sT} \frac{D^-(s)Q(s)}{D(-s^2) - \lambda N(-s^2)} \right\}$

will be nonzero only for $t > T$ and using this in (19-109)-(19-110) we conclude that $\phi_i(t)$ for $0 < t < T$ has contributions only from the first term in (19-111). Together with (19-81), finally we obtain the desired eigenfunctions to be

$$\varphi_k(t) = L^{-1} \left\{ \frac{D^+(s)P(s, \lambda_k)}{D(-s^2) - \lambda_k N(-s^2)} \right\}, \quad 0 < t < T, \quad (19-112)$$

$$\text{Re } s > \text{Re } \omega_n > 0, \quad k = 1, 2, \dots, n$$

that are orthogonal by design. Notice that in general (19-112) corresponds to a sum of modulated exponentials.

Next, we shall illustrate this procedure through some examples. First, we shall re-do Example 19.3 using the method described above.

Example 19.4: Given $R_{xx}(\tau) = e^{-\alpha|\tau|}$, we have

$$S_{xx}(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2} = \frac{N(\omega^2)}{D(\omega^2)}.$$

This gives $D^+(s) = \alpha + s$, $D^-(s) = \alpha - s$ and $P(s)$, $Q(s)$ are constants here. Moreover since $n = 1$, (19-102) reduces to $1 \pm a_1 = 0$, or $a_1 = \pm 1$ and from (19-101), ω_1 satisfies

$$e^{\omega_1 T} = \frac{D^-(\omega_1)}{D^+(\omega_1)} = \frac{\alpha - \omega_1}{\alpha + \omega_1} \quad (19-113)$$

or ω_1 is the solution of the s -plane equation

$$e^{sT} = \frac{\alpha - s}{\alpha + s} \quad (19-114)$$

But $|e^{sT}| > 1$ on the RHP, whereas $\left|\frac{\alpha-s}{\alpha+s}\right| < 1$ on the RHP. Similarly $|e^{sT}| < 1$ on the LHP, whereas $\left|\frac{\alpha-s}{\alpha+s}\right| > 1$ on the LHP.

Thus in (19-114) the solution s must be purely imaginary, and hence ω_1 in (19-113) is purely imaginary. Thus with $s = j\omega_1$ in (19-114) we get

$$e^{j\omega_1 T} = \frac{\alpha - j\omega_1}{\alpha + j\omega_1}$$

or

$$\tan(\omega_1 T / 2) = -\frac{\omega_1}{\alpha} \quad (19-115)$$

which agrees with the transcendental equation (19-65). Further from (19-108), the λ_s satisfy

$$D(-s^2) - \lambda_n N(-s^2) \Big|_{s=j\omega_n} = \alpha^2 + \omega_n^2 - 2\alpha\lambda_n = 0$$

or

$$\lambda_n = \frac{\alpha^2 + \omega_n^2}{2\alpha} > 0. \quad (19-116)$$

Notice that the λ_n in (19-66) is the inverse of (19-116) because as noted earlier λ in (19-79) is the inverse of that in (19-22).

Finally from (19-112)

$$\varphi_n(t) = L^{-1} \left\{ \frac{s + \alpha}{s^2 + \omega_n^2} \right\} = A_n \cos \omega_n t + B_n \sin \omega_n t, \quad 0 < t < T \quad (19-117)$$

which agrees with the solution obtained in (19-67). We conclude this section with a less trivial example.

Example 19.5

$$R_{xx}(\tau) = e^{-\alpha|\tau|} + e^{-\beta|\tau|}. \quad (19-118)$$

In this case

$$S_{xx}(\omega) = \frac{2\alpha}{\omega^2 + \alpha^2} + \frac{2\beta}{\omega^2 + \beta^2} = \frac{2(\alpha + \beta)(\omega^2 + \alpha\beta)}{(\omega^2 + \alpha^2)(\omega^2 + \beta^2)}. \quad (19-119)$$

This gives $D^+(s) = (s + \alpha)(s + \beta) = s^2 + (\alpha + \beta)s + \alpha\beta$. With $n = 2$, (19-107) and its companion determinant reduce to

$$\omega_2 \tanh \theta_2 = \omega_1 \tanh \theta_1$$

$$\omega_2 \coth \theta_2 = \omega_1 \coth \theta_1$$

or

$$\tanh \theta_1 = \pm \tanh \theta_2. \quad (19-120)$$

From (19-106)

$$\tanh \theta_i = \frac{(\alpha\beta + \omega_i^2) \tanh (\omega_i T / 2) + (\alpha + \beta) \omega_i}{(\alpha\beta + \omega_i^2) + (\alpha + \beta) \omega_i \tanh (\omega_i T / 2)}, \quad i = 1, 2 \quad (19-121)$$

Finally ω_1^2 and ω_2^2 can be parametrically expressed in terms of λ using (19-108) and it simplifies to

$$\begin{aligned} D(-s^2) - \lambda N(-s^2) &= s^4 - (\alpha^2 + \beta^2 - 2\lambda(\alpha + \beta))s^2 \\ &\quad + \alpha^2\beta^2 - 2\lambda(\alpha + \beta)\alpha\beta \\ &\triangleq s^4 - bs^2 + c = 0. \end{aligned}$$

This gives

$$\omega_1^2 = \frac{b(\lambda) + \sqrt{b^2(\lambda) - 4c(\lambda)}}{2}$$

and

and

$$\omega_2 = \frac{b(\lambda) - \sqrt{b^2(\lambda) - 4c(\lambda)}}{2} = \omega_1 - \sqrt{b^2(\lambda) - 4c(\lambda)}$$

and substituting these into (19-120)-(19-121) the corresponding transcendental equation for $\lambda_i s$ can be obtained. Similarly the eigenfunctions can be obtained from (19-112).