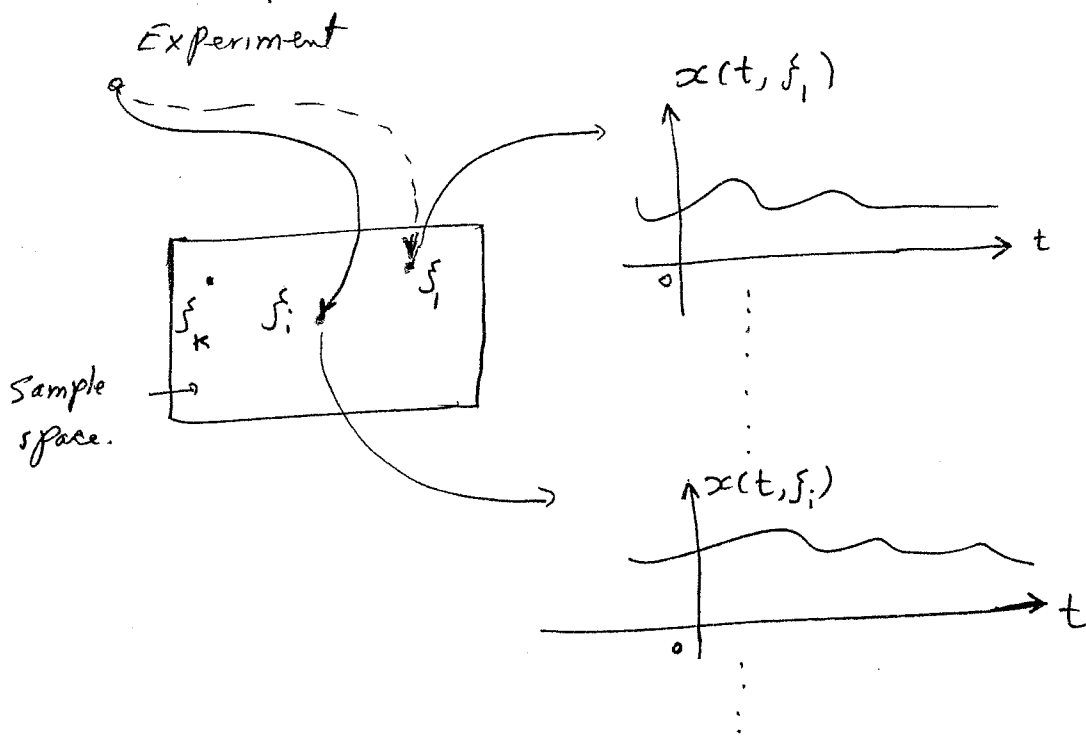


# Random Process - Appendix A, A-9 $\leftrightarrow$ A-9

Imagine a chance experiment where fens of time are assigned to each possible outcome of the experiment.

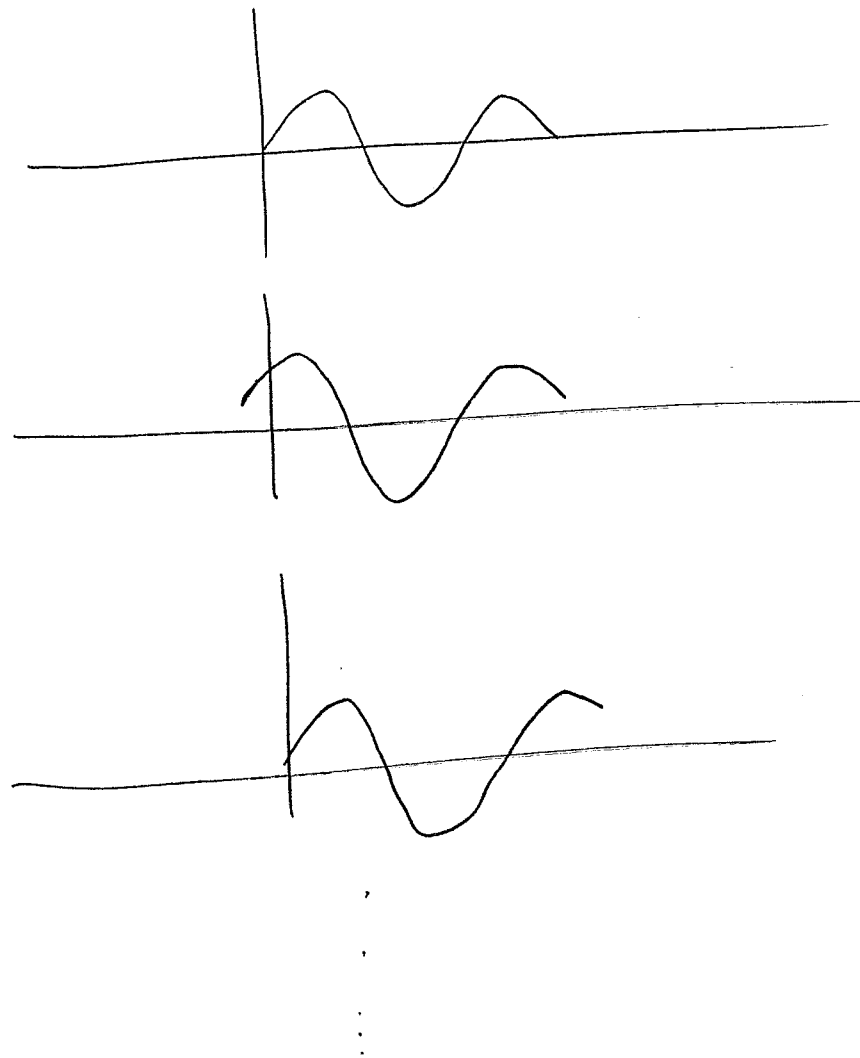


Each waveform is referred to as a sample fen. The totality of all sample fens is called an ensemble. The underlying chance experiment which gives rise to the ensemble of sample fens is called a random, or stochastic process.

Exa:  $X(t) = A \cos(\omega_0 t + \theta)$   $-\infty < t < \infty$

where  $A$  and  $\omega_0$  are constants and

$\theta$  is a r.v. varying between  $0, 2\pi$ .



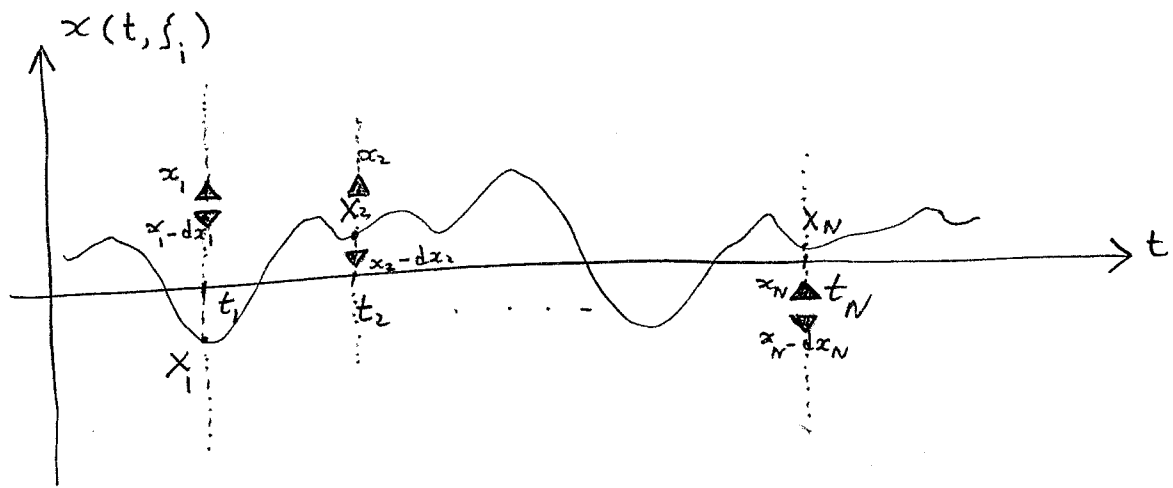
A complete description of a random process,  $x(t, \omega)$ , is given by the  $N$ -fold joint Pdf, where:

$$\int_{\mathcal{X}} f(x_1, t_1; x_2, t_2; x_3, t_3; \dots; x_N, t_N) dx_1 dx_2 \dots dx_N$$

↓ possible values that  $x(t_i)$  takes on.

$$= P(x_1 - dx_1 < X_1 \leq x_1 \text{ at } t_1, x_2 - dx_2 < X_2 \leq x_2 \text{ at } t_2, \dots$$

$$\dots, x_N - dx_N < X_N \leq x_N \text{ at } t_N) \quad (*)$$

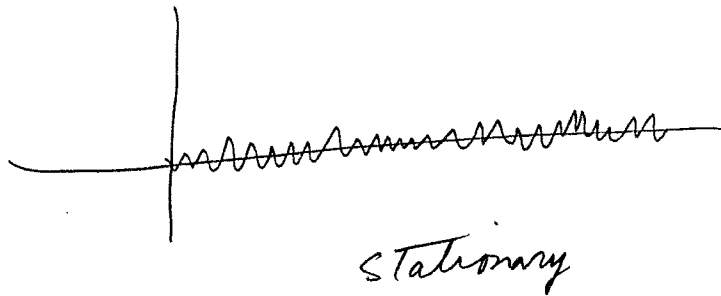
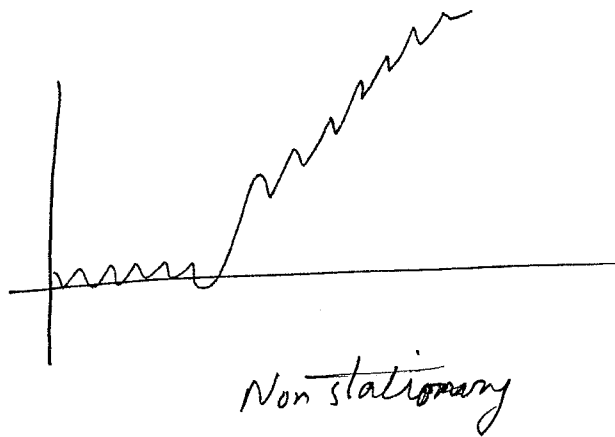


Notice that  $X_1, X_2, \dots, X_N$  are r.v.'s.

To emphasize the interpretation of (\*) Consider the relative freq interpretation which means, the joint prob\* is the # of sample yens which pass through the slits in  $N$  barriers divided by the total # <sup>$M$</sup>  of sample yens as  $M$  becomes large without bound.

It is seen that the  $N$ -fold pdf depends on  $N$  time instants  $t_1, t_2, \dots, t_N$ . In some cases, it may happen that these joint pdf's depends only on the time differences  $t_2 - t_1, t_3 - t_1, \dots, t_N - t_1$ , i.e; the choice of the time origin for the random process is immaterial. Such random processes are said to be statistically stationary.

For the stationary processes, means and variances are indep of time, and the correlation  $\rho_{\tau} =$  depends only on the time difference  $t_2 - t_1$ .



## Partial description of random processes

As in the case of r.v.'s, we may not always require a complete statistical description of a random process, or we may not be able to obtain the  $N$ -fold joint pdf even if desired. In such cases we work with various moments.

$$m(t) = E\{x(t)\} = \bar{x}(t)$$

$$\sigma^2(t) = E\{[x(t) - \bar{x}(t)]^2\}$$

$$\mu_x(t, t+\tau) = E\{[x(t) - \bar{x}(t)][x(t+\tau) - \bar{x}(t+\tau)]\}$$

Covariance

## Ergodicity:

$x(t)$  is ergodic if all its statistics can be determined from a single function  $x(t, f)$  of the process. This can also be stated as:

$x(t)$  is ergodic if time averages and ensemble averages are interchangeable.

$$m = E\{x(t)\} = \langle x(t) \rangle$$

$$\sigma^2 = E\{[x(t) - \bar{x}(t)]^2\} = \langle [x(t) - \bar{x}(t)] \rangle$$

$$R(\tau) = E\{x(t)x(t+\tau)\} = \langle x(t)x(t+\tau) \rangle$$

where

$$\langle v(t) \rangle \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T v(t) dt.$$

Notice that an ergodic process must be stationary (time is the variable of integration), but the reverse is not necessarily true.

## Properties of the Autocorrelation Fun (Stationary Process)

1.  $|R(\tau)| \leq R(0) = \text{Total Power.}$

Pf:

$$[X(t) \pm X(t+\tau)]^2 \geq 0.$$

where  $\{X(t)\}$  is stationary

$$\therefore \overbrace{X^2(t)} \pm 2 \overline{X(t)X(t+\tau)} + \overline{X^2(t+\tau)} \geq 0.$$

$$R(0) \pm 2R(\tau) + R(0) \geq 0.$$

$$\therefore -R(0) \leq R(\tau) \leq R(0)$$

2.  $R(-\tau) = R(\tau)$

$$\begin{aligned} \text{Pf: } R(\tau) &= \overline{X(t)X(t+\tau)} = \overline{X(t'-\tau)X(t')} \\ &= R(-\tau) \\ t' &= t+\tau. \end{aligned}$$



$$3. \lim_{|T| \rightarrow \infty} R(T) = \overline{X(t)^2} \leftarrow \text{dc Power}$$

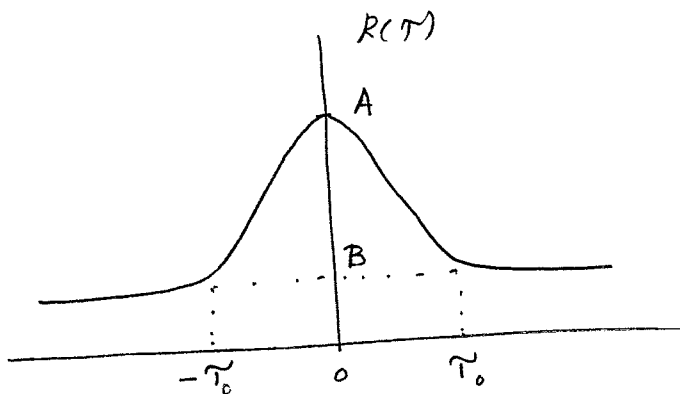
$$\begin{aligned} \text{Pf: } \lim_{|T| \rightarrow \infty} R(T) &= \lim_{|T| \rightarrow \infty} \overline{X(t)X(t+T)} \cong \overline{X(t)} \overline{X(t+T)} \\ &= \overline{X(t)^2} = \text{dc Power.} \end{aligned}$$

$$4. S(f) \xleftrightarrow{F} R(T)$$

$$\text{or } S(f) = F\{R(T)\}$$

The function  $S(f)$  is called the power spectral density of  $X(t)$ . Notice that since  $R(T)$  is real and even therefore  $S(f)$  is real and even.

Exn:



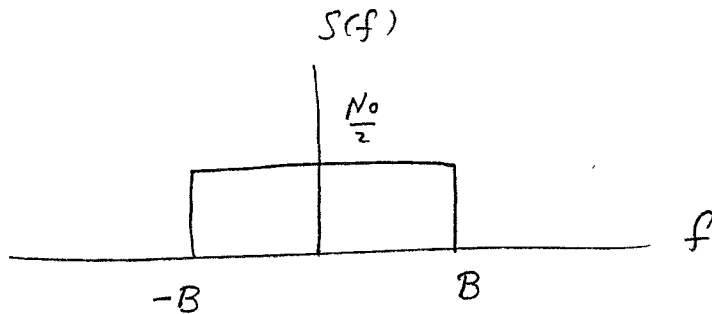
$$S(f) = \int_{-\infty}^{\infty} R(T) e^{j\omega T} dT$$

real  $\rightarrow$   
 even  $\rightarrow S(f) = S(-f)$

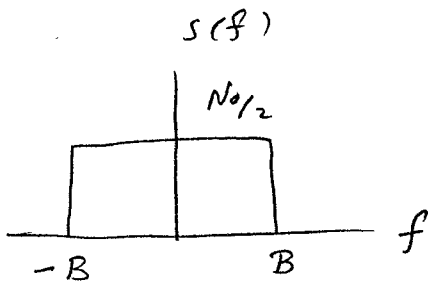
1. dc Power =  $R(\pm\infty) = B$
2. Total Power =  $R(0) = A$
3. ac power =  $A - B = \sigma^2$

Exa: processes for which

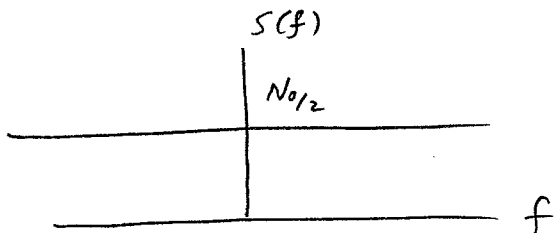
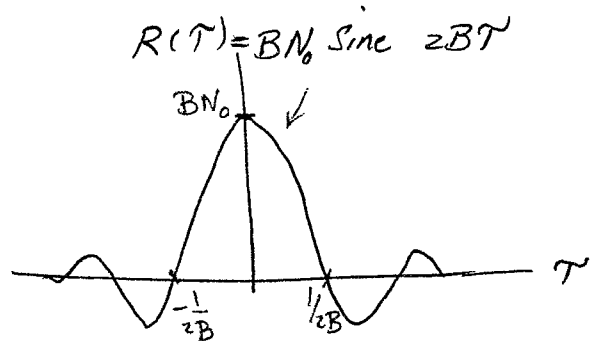
$$S(f) = \begin{cases} \frac{N_0}{2} & |f| \leq B \\ 0 & \text{elsewhere.} \end{cases}$$



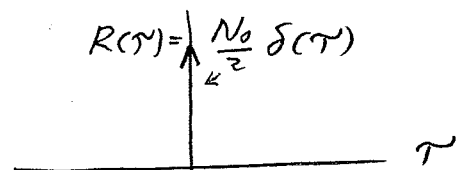
are commonly referred to as bandlimited white since, as  $B \rightarrow \infty$ , all frequencies are present, in which case the process is simply called white. notice that:



$\longleftrightarrow$   
F



$\longleftrightarrow$   
F



infinite power!

## Cross-Correlation of Two Processes

$$R_{12}(t_1, t_2) = E\{x(t_1)y(t_2)\}.$$

For stationary processes:

$$R_{12}(\tau) = E\{x(t)y(t+\tau)\}$$

Properties:

1.  $R_{12}(\tau) = R_{21}(-\tau)$

Pf:  $R_{12}(\tau) = E\{x(t)y(t+\tau)\}$

Let  $t' = t + \tau$ :

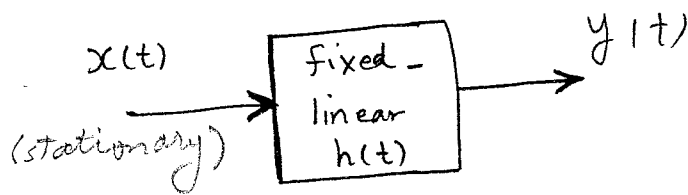
$$= E\{x(t-\tau)y(t')\}$$

$$= E\{y(t')x(t-\tau)\} = R_{21}(-\tau).$$

2. Cross-power spectral density of two stationary processes is defined as follows:

$$S_{12}(f) = \mathcal{F}\{R_{12}(\tau)\}.$$

# linear systems and Random Processes.



$$y(t) = \int_{-\infty}^{\infty} h(\alpha) x(t-\alpha) d\alpha.$$

$$m_y = E\{y(t)\} = \int_{-\infty}^{\infty} h(\alpha) \underbrace{E\{x(t-\alpha)\}}_{m_x} d\alpha$$

$$\therefore m_y = m_x \int_{-\infty}^{\infty} h(\alpha) d\alpha.$$

$$R_y(\tau) = E\{y(t) y(t+\tau)\}$$

$$= E \left[ \int_{-\infty}^{\infty} h(\alpha) x(t-\alpha) d\alpha \int_{-\infty}^{\infty} h(\beta) x(t+\tau-\beta) d\beta \right]$$

$$= \iint_{-\infty}^{\infty} h(\alpha) h(\beta) \underbrace{E\{x(t-\alpha) x(t+\tau-\beta)\}}_{R_x(\tau-\beta+\alpha)} d\beta d\alpha$$

(1)

The last eq<sup>n</sup> appears difficult to evaluate.

Let us consider the freq domain.

Know  $R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f \tau} df$

$$\therefore R_x(\tau - \beta + \alpha) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f(\tau - \beta + \alpha)} df \quad (2)$$

(2) in (1)  $\Rightarrow$

$$R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) h(\beta) \left[ \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f(\tau - \beta + \alpha)} df \right] d\beta d\alpha.$$

$$= \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f \tau} df \left[ \underbrace{\int_{-\infty}^{\infty} h(\alpha) e^{j2\pi f \alpha} d\alpha}_{H^*(f)} \underbrace{\int_{-\infty}^{\infty} h(\beta) e^{-j2\pi f \beta} d\beta}_{H(f)} \right]$$

$$= \int_{-\infty}^{\infty} \underbrace{H(f) H^*(f)}_{|H(f)|^2} S_x(f) e^{j2\pi f \tau} df = \mathcal{F}^{-1} \left\{ |H(f)|^2 S_x(f) \right\}$$

$$\therefore S_y(f) = |H(f)|^2 S_x(f) \quad *$$

also note that

$$\underbrace{R_x(\tau)}_{\sigma_x^2 + \text{dc power}} \Big|_{\tau=0} = E\{x^2(t)\} = \int_{-\infty}^{\infty} S_x(f) df \quad *$$

↙ Total Power.

Now let us find  $R_{xy}(\tau)$ . Know;

$$R_{xy}(\tau) = E\{x(t)y(t+\tau)\}$$

$$\text{But } y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)du$$

$$\therefore R_{xy}(\tau) = E\left\{x(t) \int_{-\infty}^{\infty} h(u)x(t+\tau-u)du\right\}$$

$$= \int_{-\infty}^{\infty} h(u) \underbrace{E\{x(t)x(t+\tau-u)\}}_{R_x(\tau-u)} du$$

$$\therefore R_{xy}(\tau) = h(\tau) * R_x(\tau) \quad *$$

taking the  $F\{\cdot\}$  of both sides of the last eq<sup>n</sup> yields:

$$S_{xy}(f) = H(f) S_x(f) \quad *$$

Also,

$$R_{yx}(\tau) = R_{xy}(-\tau) = h(-\tau) * \underbrace{R_x(-\tau)}_{R_x(\tau)}$$

$$\therefore R_{yx}(\tau) = h(-\tau) * R_x(\tau) \quad *$$

$$\therefore S_{yx}(f) = H^*(f) S_x(f) \quad *$$

notice that

$$S_{xy}^*(f) = H^*(f) \underbrace{S_x^*(f)}_{\substack{\text{always real} \\ \text{because } R_x(\tau) \text{ is real and} \\ \text{even.}}} = S_{yx}(f)$$

$$\therefore S_{xy}^*(f) = S_{yx}(f) \quad *$$

In the following, we summarize the important relationships for fixed linear system:

$$S_y(f) = |H(f)|^2 S_x(f)$$

$$R_y(\tau) = \int_{-\infty}^{\infty} |H(f)|^2 S_x(f) e^{j2\pi f \tau} df$$

$$R_{xy}(\tau) = h(\tau) * R_x(\tau)$$

$$S_{xy}(f) = H(f) S_x(f)$$

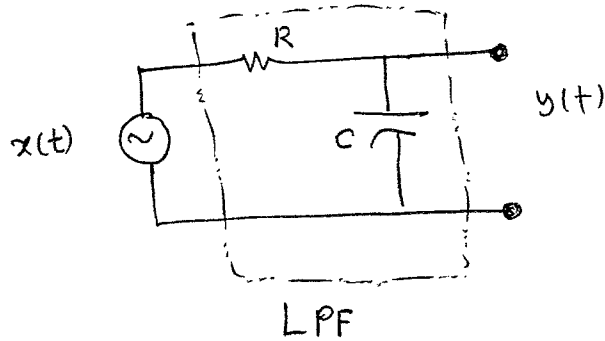
$$R_{yx}(\tau) = h(-\tau) * R_x(\tau)$$

$$S_{yx}(f) = H^*(f) S_x(f)$$

$$S_{xy}^*(f) = S_{yx}(f)$$



EXA : Consider a LPF as shown below:



Let  $x(t)$  be a sample function from white, Gaussian process with  $S_x(f) = \frac{N_0}{2}$  ;  $|f| < \infty$ .

$$\therefore S_y(f) = |H(f)|^2 S_x(f)$$

$$H(f) = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$|H(f)| = \frac{1}{\sqrt{1 + (f \cdot 2\pi R C)^2}}$$

$$= \frac{1}{1 + j\omega R C}$$

$$\therefore S_y(f) = \frac{N_0/2}{1 + (2\pi R C f)^2}$$

$$R_y(\tau) = \frac{N_0}{4RC} e^{-|\tau|/RC}$$

$$m_y^2 = \lim_{\tau \rightarrow \pm\infty} R_y(\tau) \quad \text{When } \tau \rightarrow \pm\infty \Rightarrow m_y^2 = 0.$$

$$\sigma_y^2 = R_y(0) = \frac{N_0}{4RC}$$

Since the input is Gaussian, so is the output.

$$f_y(y, t) = \frac{1}{\sqrt{2\pi} \left(\frac{N_0}{4R_c}\right)} e^{-y^2 / \frac{N_0}{4R_c}}, \quad \forall t.$$

$f_y(y)$  ← 1st order PDF.

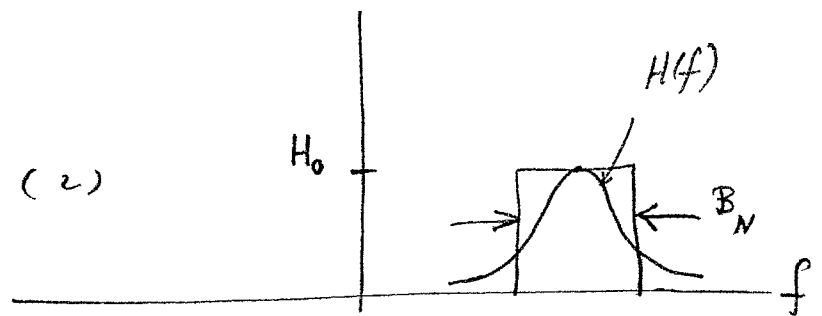
## Noise - Equivalent Bandwidth

If we pass white noise through a filter with transfer function  $H(f)$ , the avg power at the output is

$$P_y = \int_{-\infty}^{\infty} S_y(f) df = \int_{-\infty}^{\infty} \frac{N_0}{2} |H(f)|^2 df = N_0 \int_0^{\infty} |H(f)|^2 df \quad (1)$$

If the filter were ideal with BW  $B_N$  and midband gain  $H_0$ , the noise power at the output of filter would be

$$P_y = N_0 B_N H_0^2 \quad (2)$$



the question is now:

What is the BW of an ideal, fictitious filter with the same midband gain as  $H(f)$  that passes the same noise power?

$$(1), (2) \Rightarrow B_N = \frac{1}{H_0^2} \int_0^{\infty} |H(f)|^2 df$$

$B_N$  is called the noise-equivalent BW of  $H(f)$ .

Aside

## Cyclostationary random processes

A random process is referred to as Cyclostationary if  $E\{x(t)\}$  and  $R_x(t_1, t_2)$  are both periodic in time.

$$\therefore E\{x(t)\} = E\{x(t+T)\}$$

$$R_x(t_1, t_2) = R_x(t_1+T, t_2+T)$$

A way to remove the nonstationary character of the r.p. is through "phase randomizing".

Exa Let  $x(t) = A \cos \omega_0 t$ ,  $A$  is r.v. and

$\omega_0$  is constant. Then

$$E\{x(t)\} = E\{A\} \cos \omega_0 t$$

$$R_x(t_1, t_2) = E\{A \cos \omega_0 t_1, A \cos \omega_0 t_2\} = \sigma_A^2 \cos \omega_0 t_1 \cos \omega_0 t_2.$$

$\therefore x(t)$  is cyclostationary.

Now, consider  $X(t) = A \cos(\omega_0 t + \theta)$  where  $\theta$  is a uniform r.v. in the interval  $(0, 2\pi)$ .

$$\therefore E\{X(t)\} = E\{A\} E\{\cos(\omega_0 t + \theta)\} = 0.$$

$$R_X(t_1, t_2) = E\{A^2 \cos(\omega_0 t_1 + \theta) \cos(\omega_0 t_2 + \theta)\}$$

$$= \frac{1}{2} E\{A^2\} E\left[\cos(\omega_0(t_1 - t_2)) + \cos(\omega_0(t_1 + t_2) + 2\theta)\right]$$

$$= \frac{1}{2} E\{A^2\} \cos \omega_0(t_1 - t_2)$$

thus by allowing a random phase <sup>in the r.p.</sup>, we made the process stationary!

$$\begin{aligned} & E\{\cos(\omega_0 t + \theta)\} \\ &= \int_0^{2\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\omega_0 t + \theta) d\theta = 0. \end{aligned}$$

Exa. Let

$$x(t) = \sum_{k=-\infty}^{\infty} A_k P(t - kT - \Delta) \quad \leftarrow \text{PAM signal}$$

Constant  
↓ ↓  
↑  
pulse of  
indep. arbitrary  
r.v. shaper.

$$\begin{aligned} \therefore E\{x(t)\} &= \sum_{k=-\infty}^{\infty} E\{A_k\} P(t - kT - \Delta) \\ &= a \sum_{k=-\infty}^{\infty} P(t - kT - \Delta). \end{aligned}$$

$$R_x(t+T, t) = E\left\{ \sum_k \sum_j A_k A_j P(t - kT - \Delta) P(t + T - jT - \Delta) \right\}$$

let  $j = k'$   
 $k = k' + m$

Let  $E\{A_k A_{k+m}\} = \alpha_{|m|}$

$$\therefore R_x(t+T, t) = \sum_{m=-\infty}^{\infty} \alpha_{|m|} \sum_{k'=-\infty}^{\infty} P(t - (k'+m)T - \Delta) P(t + T - k'T - \Delta)$$

Replacing  $t$  by  $t+T$  in  $E\{x(t)\}$  and  $R_x(t+T, t)$

Shows that this process is Cyclostationary.

Now, let  $\Delta$  be uniformly distributed in  $(-\frac{T}{2}, \frac{T}{2})$ .

then

$$E\{x(t)\} = a \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=-\infty}^{\infty} P(t-kT-\delta) \frac{P(\delta)}{\Delta} d\delta$$

$$= \frac{a}{T} \sum_{k=-\infty}^{\infty} \int_{t-[k-\frac{1}{2}]T}^{t-[k+\frac{1}{2}]T} P(u) du = \frac{a}{T} \int_{-\infty}^{\infty} P(t) dt \leftarrow \text{indep of } t.$$

$$R_x(t+\tau, t) = \sum_{k,m=-\infty}^{\infty} \alpha_{|m|} \int_{-\frac{T}{2}}^{\frac{T}{2}} P(t+\tau-kT-\delta) P(t-(k+m)T-\delta) \frac{d\delta}{T}$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \alpha_{|m|} \sum_{k=-\infty}^{\infty} \int_{t-[k-\frac{1}{2}]T}^{t-[k+\frac{1}{2}]T} P(u+\tau) P(u-mT) du$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \alpha_{|m|} \int_{-\infty}^{\infty} P(t'+\tau+mT) P(t') dt'$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \alpha_{|m|} r(\tau+mT) \leftarrow \text{indep of } t.$$

where

$$r(\tau) = \int_{-\infty}^{\infty} P(t+\tau) P(t) dt.$$