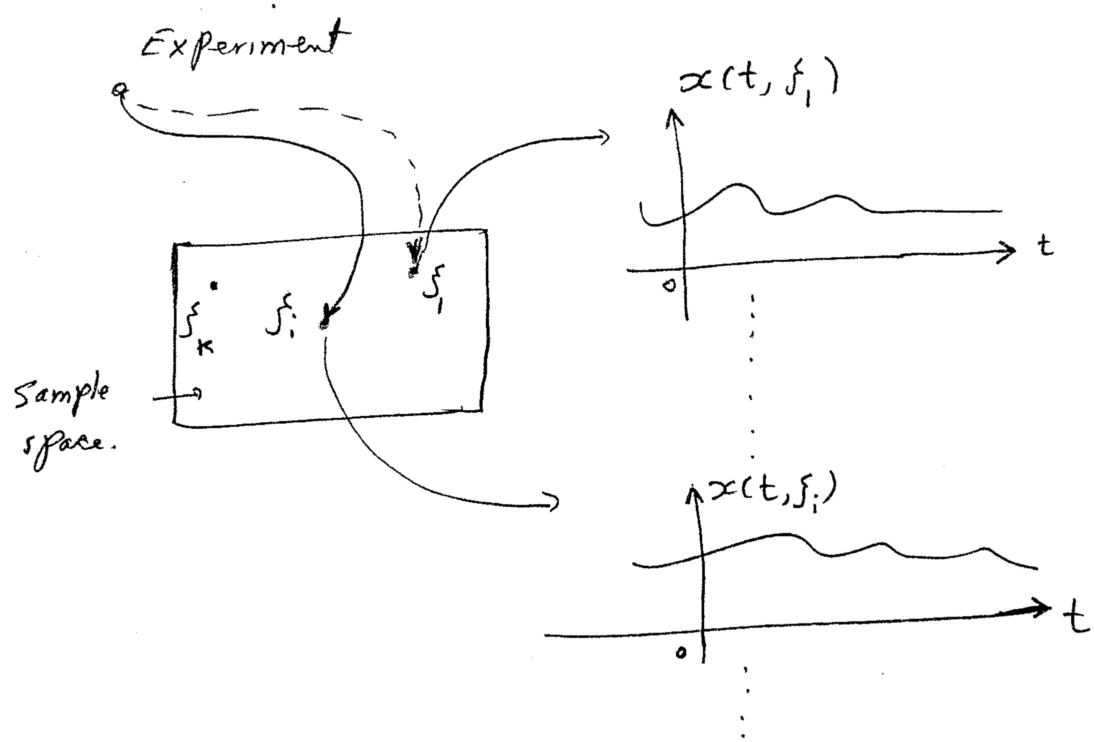
Random Process - Appendix A, A-9 ⟷ A-9

Imagine a chance experiment where fcns of time are assigned to each possible outcome of the experiment.



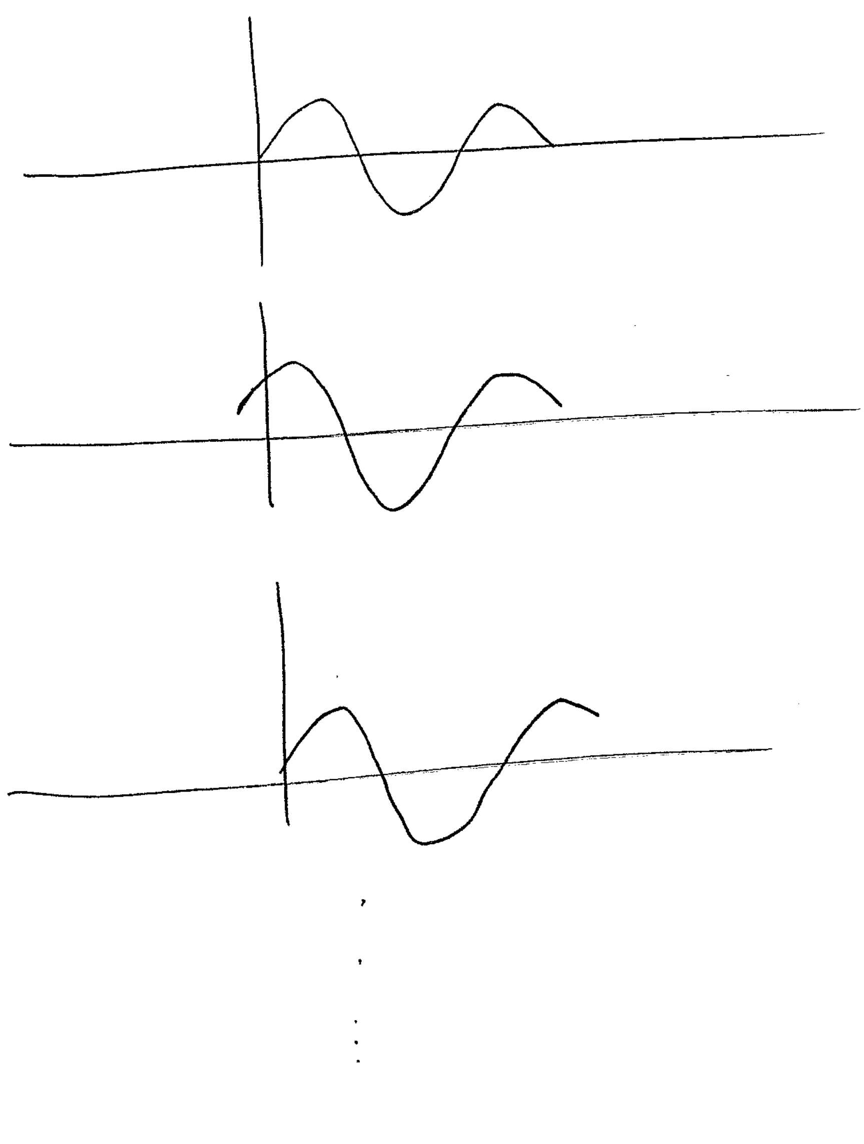
Each waveform is referred to as a sample fcn.

The totality of all sample fcns is called an ensemble. The underlying chance experiment which gives rise to the ensemble of sample fcns is called a random, or stochastic process.

Exa: $X(t) = A \cos(\omega_0 t + \theta) \qquad -\infty < t < \infty$

where A and ω_0 are constants and

θ is a r.v varying between $0, 2\pi$.



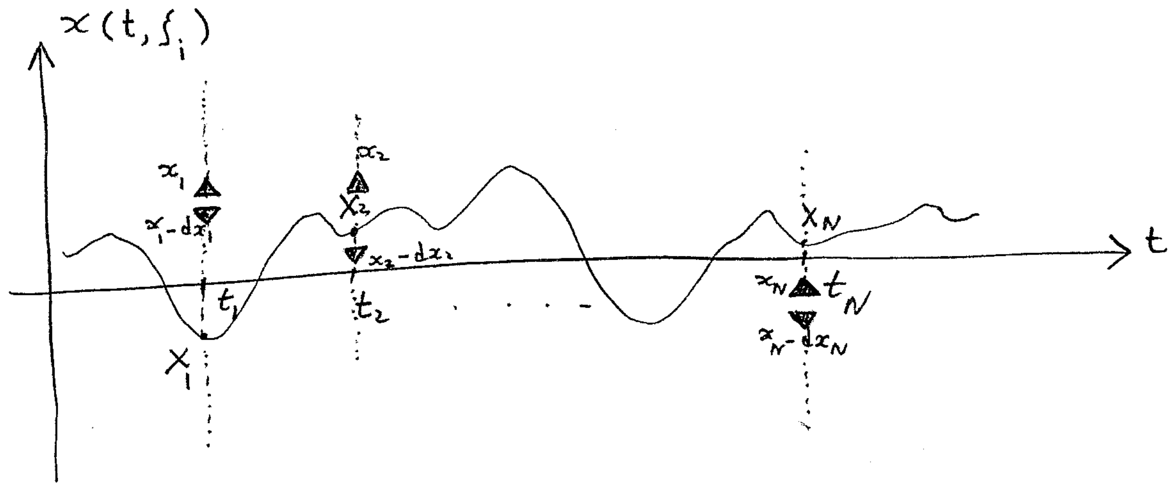
A complete description of a random process, $x(t, \varsigma)$, is given by the N-fold joint Pdf, where:

$$f_X(x_1, t_1 ; x_2, t_2 ; x_3, t_3 ; \cdots ; x_N, t_N)\, dx_1\, dx_2 \cdots dx_N$$

possible values that $x(t_1)$ takes on.

$$= P(x_1 - dx_1 < X_1 \le x_1 \text{ at } t_1,\ x_2 - dx_2 < X_2 \le x_2 \text{ at } t_2, \cdots$$

$$\cdots,\ x_N - dx_N < X_N \le x_N \text{ at } t_N). \quad (*)$$

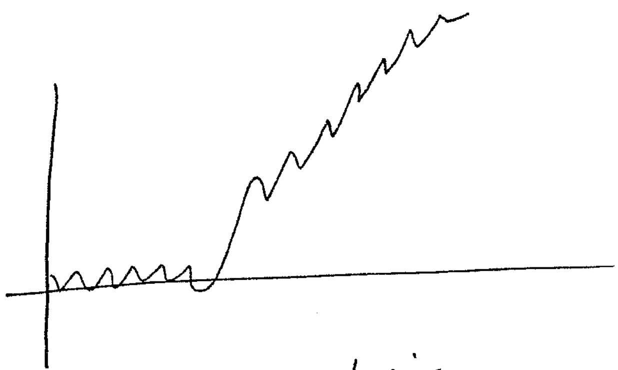


Notice that $X_1, X_2, \cdots, X_N$ are r.v's.

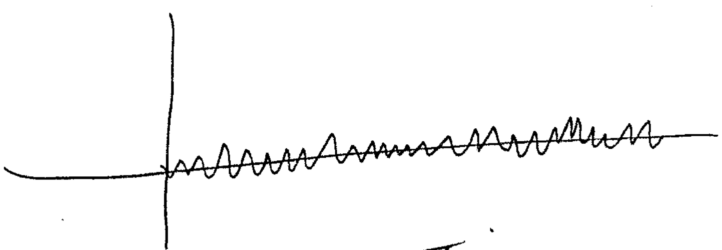
To emphasize the interpretation of (*) consider the relative freq interpretation which means, the joint prob* is the # of sample fcns which pass through the slits in N barriers divided by the total # of M sample fcns as M becomes large without bound.

It is seen that the N-fold pdf depends on N time instants $t_1, t_2, \ldots, t_N$. In some cases, it may happen that these joint pdf's depends only on the time differences $t_2 - t_1$, $t_3 - t_1$, ..., $t_N - t_1$, i.e; the choice of the time origin for the random process is immaterial. Such random processes are said to be <u>statistically stationary</u>.

For the stationary processes, means and variances are indep of time, and the correlation coef^{n} depends only on the time difference $t_2 - t_1$.



Non stationary



Stationary

Partial description of random processes

As in the case of r.v's, we may not always require a complete statistical description of a random process, or we may not be able to obtain the N-fold joint pdf even if desired. In such cases we work with various moments.

$$m(t) = E\{x(t)\} = \overline{x(t)}$$

$$\sigma^2(t) = E\{[x(t) - \overline{x(t)}]^2\}$$

$$\underbrace{\mu_x(t, t+\tau)}_{\text{Covariance}} = E\{[x(t) - \overline{x(t)}][x(t+\tau) - \overline{x(t+\tau)}]\}$$

Ergodicity:

$X(t)$ is ergodic if all its statistics can be determined from a single function $X(t,\xi)$ of the process. This can also be stated as:

$X(t)$ is ergodic if time averges and ensemble averages are interchangeable.

$$m = E\{X(t)\} = \langle x(t) \rangle$$

$$\sigma^2 = E\{[X(t) - \overline{X(t)}]^2\} = \langle [X(t) - \overline{X(t)}] \rangle$$

$$R(\tau) = E\{X(t)X(t+\tau)\} = \langle x(t)X(t+\tau) \rangle$$

where

$$\langle v(t) \rangle \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t)\,dt.$$

Notice that an ergodic process must be stationary (time is the variable of integration), but the reverse is not necessarily true.

Properties of the Autocorrelation fcn (stationary process)

1. $|R(\tau)| \leq R(0) = \text{Total Power}$.

Pf:

$$[X(t) \pm X(t+\tau)]^2 \geq 0.$$

where $\{X(t)\}$ is stationary

$$\therefore \quad \overline{X^2(t)} \pm 2\overline{X(t)X(t+\tau)} + \overline{X^2(t+\tau)} \geq 0$$

$$R(0) \pm 2R(\tau) + R(0) \geq 0.$$

$$\therefore \quad -R(0) \leq R(\tau) \leq R(0)$$

2. $R(-\tau) = R(\tau)$

Pf:

$$R(\tau) = \overline{X(t)X(t+\tau)} = \overline{X(t'-\tau)X(t')}$$

$$= R(-\tau)$$

$t' = t + \tau$.

3. $\lim\limits_{|\tau|\to\infty} R(\tau) = \overline{X(t)}^{2}$ $\leftarrow$ dc Power

Pf: $\lim\limits_{|\tau|\to\infty} R(\tau) = \lim\limits_{|\tau|\to\infty} \overline{X(t)X(t+\tau)} \cong \overline{X(t)}\;\overline{X(t+\tau)}$

$$= \overline{X(t)}^{2} = \text{dc Power.}$$

4. $$S(f) \xleftrightarrow{\mathcal{F}} R(\tau)$$

or $S(f) = \mathcal{F}\{R(\tau)\}$

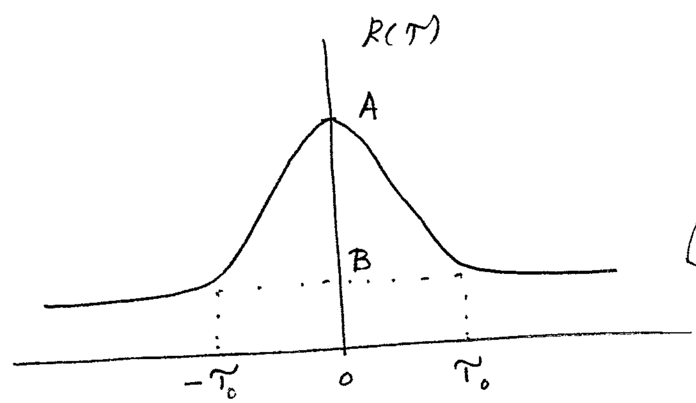
The function $S(f)$ is called the power spectral density of $x(t)$. Notice that since $R(\tau)$ is real and even Therefore $S(f)$ is real and even.

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

real

even $\rightarrow$ $S(f) = S(-f)$

Exa:



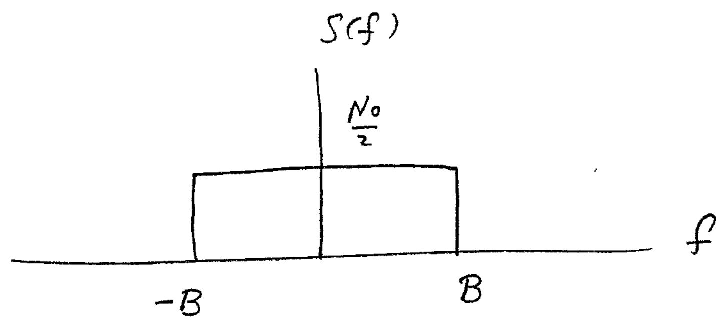
1. dc Power $= R(\pm\infty) = B$

2. Total Power $= R(0) = A$

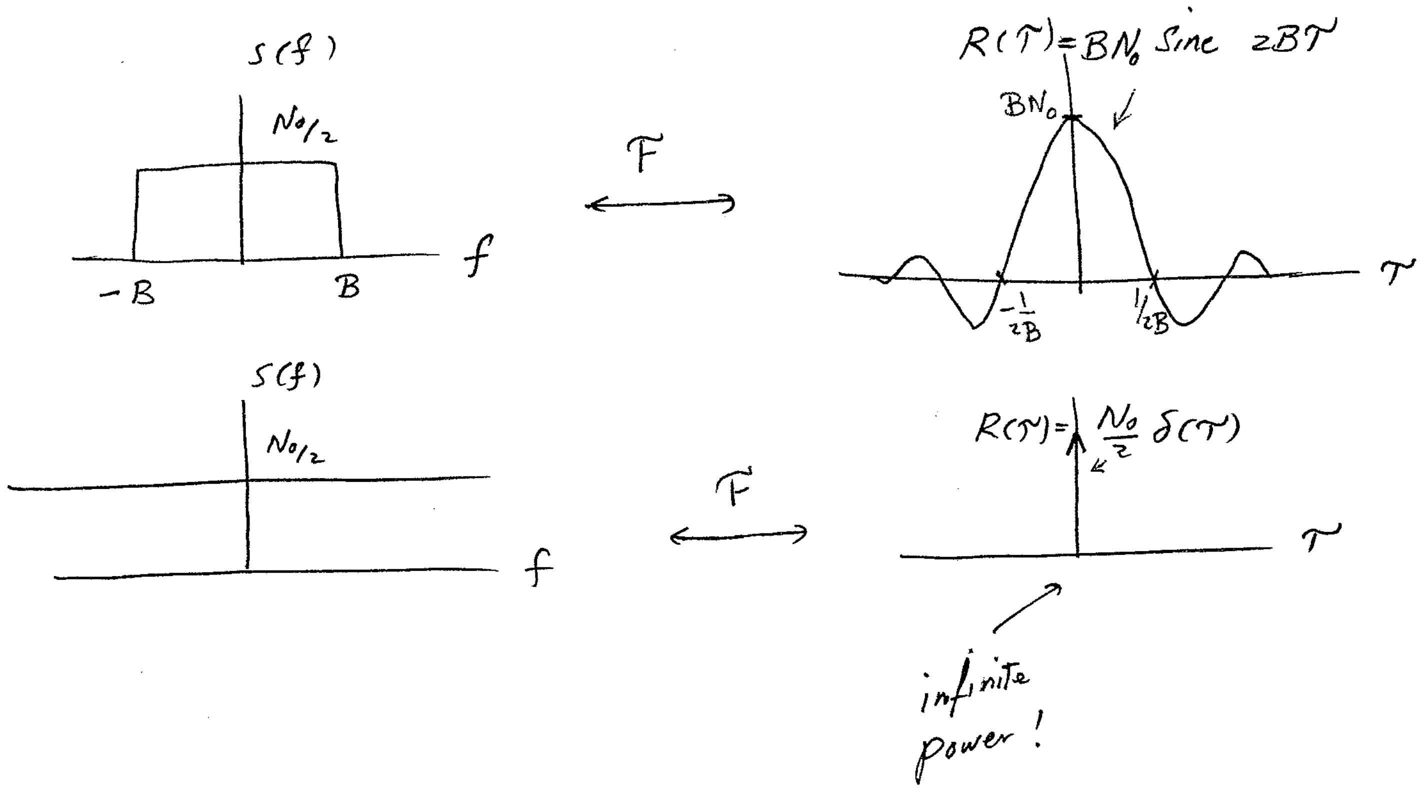
3. ac power $= A - B = \sigma^2$

Exa: processes for which

$$S(f) = \begin{cases} \frac{N_0}{2} & |f| \le B \\ 0 & \text{elsewhere.} \end{cases}$$



are commonly referred to as bandlimited white since, as $B \to \infty$, all freqs are present, in which case the process is simply called <u>white</u>. Notice that:



Cross-Correlation of two Processes

$$R_{12}(t_1, t_2) = E\{ x(t_1) y(t_2) \}.$$

For stationary processes:

$$R_{12}(\tau) = E\{ x(t)\, y(t+\tau) \}$$

Properties:

1. $R_{12}(\tau) = R_{21}(-\tau)$

Pf: $R_{12}(\tau) = E\{ x(t) y(t+\tau) \}$ Let $t' = t+\tau$:

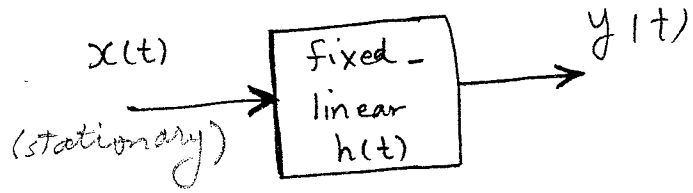
$$= E\{ x(t'-\tau)\, y(t') \}$$

$$= E\{ y(t')\, x(t'-\tau) \} = R_{21}(-\tau).$$

2. Cross-power spectral density of two stationary processes is defined as follows:

$$S_{12}(f) \triangleq \mathcal{F}\{ R_{12}(\tau) \}.$$

Linear systems and Random processes.



$$y(t) = \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha .$$

$$m_y = E\{y(t)\} = \int_{-\infty}^{\infty} h(\alpha)\, \underbrace{E\{x(t-\alpha)\}}_{m_x}\, d\alpha$$

$$\therefore \quad m_y = m_x \int_{-\infty}^{\infty} h(\alpha)\, d\alpha .$$

$$R_y(\tau) = E\{y(t)\, y(t+\tau)\}$$

$$= E\left[\int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha \int_{-\infty}^{\infty} h(\beta)\, x(t+\tau-\beta)\, d\beta \right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha) h(\beta)\, \underbrace{E\{x(t-\alpha)\, x(t+\tau-\beta)\}}_{R_x(\tau-\beta+\alpha)}\, d\beta\, d\alpha \qquad (1)$$

the last eq^n appears difficult to evaluate.

let us consider the freq domain.

Know $$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f \tau} df$$

$$\therefore \quad R_X(\tau-\beta+\alpha) = \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f(\tau-\beta+\alpha)} df \qquad (2)$$

(2) in (1) $\Rightarrow$

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha) h(\beta) \left[\int_{-\infty}^{\infty} S_X(f) e^{j2\pi f(\tau-\beta+\alpha)} df\right] d\beta\, d\alpha.$$

$$= \int_{-\infty}^{\infty} S_X(f)\, e^{j2\pi f \tau} df \left[\underbrace{\int_{-\infty}^{\infty} h(\alpha) e^{j2\pi f \alpha} d\alpha}_{H^*(f)} \underbrace{\int_{-\infty}^{\infty} h(\beta) e^{-j2\pi f \beta} d\beta}_{H(f)}\right]$$

$$= \int_{-\infty}^{\infty} \underbrace{H(f) H^*(f)}_{|H(f)|^2} S_X(f) e^{j2\pi f \tau} df. = \mathcal{F}^{-1}\left\{ |H(f)|^2 S_X(f) \right\}$$

$$\therefore \quad S_y(f) = |H(f)|^2 S_x(f) \qquad *$$

also note that

total power.

$$\underbrace{R_x(\tau)\Big|_{\tau=0}}_{\sigma_x^2 + \text{dc power}} = E\{x^2(t)\} = \int_{-\infty}^{\infty} S_x(f)\, df \qquad *$$

Now let us find $R_{xy}(\tau)$. Know;

$$R_{xy}(\tau) = E\{x(t)\, y(t+\tau)\}$$

But $y(t) = \int_{-\infty}^{\infty} h(u)\, x(t-u)\, du$

$$\therefore \quad R_{xy}(\tau) = E\left\{ x(t) \int_{-\infty}^{\infty} h(u)\, x(t+\tau-u)\, du \right\}$$

$$= \int_{-\infty}^{\infty} h(u) \underbrace{E\{x(t)\, x(t+\tau-u)\}}_{R_x(\tau-u)}\, du$$

$\therefore$

$$R_{xy}(\tau) = h(\tau) * R_x(\tau) \qquad *$$

taking the $\mathcal{F}\{\cdot\}$ of both sides of the last eq$\underline{n}$ yields:

$$S_{xy}(f) = H(f)\; S_x(f) \qquad *$$

Also,

$$R_{yx}(\tau) = R_{xy}(-\tau) = h(-\tau) * \underbrace{R_x(-\tau)}_{R_x(\tau)}$$

$$\therefore \quad R_{yx}(\tau) = h(-\tau) * R_x(\tau) \qquad *$$

$$\therefore \quad S_{yx}(f) = H^*(f)\, S_x(f) \qquad *$$

notice that

$$S_{xy}^*(f) = H^*(f) \underbrace{\cancel{S_x^*(f)}}_{\text{always real}}^{S_x(f)} = S_{yx}(f)$$

always real because $R_x(\tau)$ is real and even.

$$\therefore \quad S_{xy}^*(f) = S_{yx}(f) \qquad *$$

In the following, we summarize the important relationships for fixed linear system:

$$S_y(f) = |H(f)|^2 S_x(f)$$

$$R_y(\tau) = \int_{-\infty}^{\infty} |H(f)|^2 S_x(f)\, e^{j2\pi f\tau}\, df$$

$$R_{xy}(\tau) = h(\tau) * R_x(\tau)$$

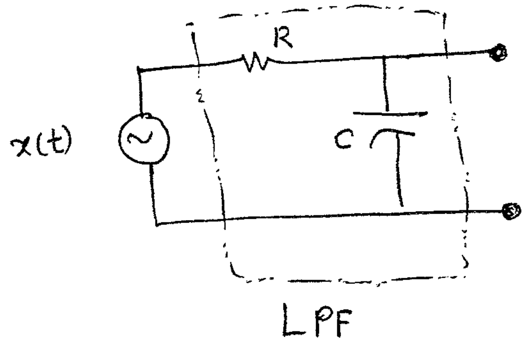
$$S_{xy}(f) = H(f)\, S_x(f)$$

$$R_{yx}(\tau) = h(-\tau) * R_x(\tau)$$

$$S_{yx}(f) = H^*(f)\, S_x(f)$$

$$S_{xy}^*(f) = S_{yx}(f)$$

Exa : Consider a LPF as shown below:



Let $x(t)$ be a sample function from white, Gaussian process with $S_x(f) = \frac{N_o}{2}$; $|f| < \infty$.

$$\therefore \quad S_y(f) = |H(f)|^2 S_x(f)$$

$$H(f) = \frac{1/j\omega c}{R + \frac{1}{j\omega c}} = \frac{1}{1 + j\underset{2\pi f}{\omega} RC}$$

$$|H(f)| = \frac{1}{\sqrt{1+(f\ 2\pi RC)^2}}.$$

$$\therefore \quad S_y(f) = \frac{N_o/2}{1+(2\pi RC f)^2}$$

$$R_y(\tau) \overset{=\mathcal{F}^{-1}\{S_y(f)\}}{=} \frac{N_o}{4RC} e^{-|\tau|/RC}$$

$m_y^2 = \lim R_y(\tau)$ when $\tau \to \pm\infty \Rightarrow m_y^2 = 0.$

$$\sigma_y^2 = R_y(0) = \frac{N_o}{4RC}.$$

Since the input is Gaussian, so is the output.

$$\underbrace{f_y(y,t)}_{f_y(y)} = \frac{1}{\sqrt{2\pi\left(N_0/4RC\right)}}\, e^{-y^2 \big/ \frac{N_0}{4RC}}, \quad \forall t.$$

1st order PDF.

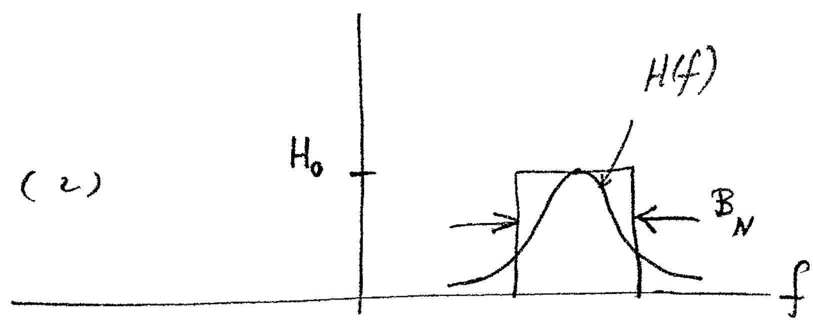
Noise - Equivalent Bandwidth

If we pass white noise through a filter with transfer fcn $H(f)$, the avg power at the output is

$$P_y = \int_{-\infty}^{\infty} S_y(f)\,df = \int_{-\infty}^{\infty} \frac{N_0}{2} |H(f)|^2\,df = N_0 \int_0^{\infty} |H(f)|^2\,df \qquad (1)$$

If the filter were ideal with BW B_N and midband gain H_0, the noise power at the output of filter would be

$$P_y = N_0 B_N H_0^2 \qquad (2)$$



the question is now:

What is the BW of an ideal, fictitious filter with the same midband gain as $H(f)$ that passes the same noise power?

(1), (2) $\Rightarrow$ $$B_N = \frac{1}{H_0^2} \int_0^{\infty} |H(f)|^2\,df .$$

B_N is called the noise-equivalent BW of $H(f)$.

Aside Cyclostationry random processes

A random process is referred to as Cyclostationry if $E\{X(t)\}$ and $R_X(t_1, t_2)$ are both periodic in time.

$$\therefore \quad E\{X(t)\} = E\{X(t+T)\}$$

$$R_X(t_1, t_2) = R_X(t_1+T, t_2+T) .$$

A way to remove the nonstationry character of the r.p. is through "phase randomizing".

Exa Let $X(t) = A \cos w_0 t$, A is r.v and w_0 is constanT. then

$$E\{X(t)\} = E\{A\} \cos w_0 t$$

$$R_X(t_1,t_2) = E\{A \cos w_0 t_1 \; A \cos w_0 t_2\} = \sigma_A^2 \cos w_0 t_1 \cos w_0 t_2 .$$

$\therefore$ $X(t)$ is Cyclo stationry.

Now, consider $X(t) = A \cos(w_0 t + \theta)$ where θ is a uniform r.v. in the interval $(0, 2\pi)$.

$$\therefore \quad E\{X(t)\} = E\{A\} E\{\cos(w_0 t + \theta)\} = 0.$$

$$R_X(t_1, t_2) = E\{A^2 \cos(w_0 t_1 + \theta) \cos(w_0 t_2 + \theta)\}$$

$$= \frac{1}{2} E\{A^2\} E\left[\cos(w_0(t_1 - t_2)) + \cos(w_0(t_1 + t_2) + 2\theta)\right].$$

$$= \frac{1}{2} E\{A^2\} \cos w_0 (t_1 - t_2).$$

thus by allowing a random phase in the r.p., we made the process stationary!

$$E\{\cos(w_0 t + \theta)\}$$

$$= \int_0^{2\pi} \cos(w_0 t + \theta) \underbrace{f_\theta(\theta)}_{\frac{1}{2\pi}} d\theta$$

$$= \frac{1}{2\pi} \sin(w_0 t + \theta) \Big|_0^{2\pi} = 0.$$

Exa. Let

Constant

$$x(t) = \sum_{k=-\infty}^{\infty} A_k \, p(t - kT - \Delta) \quad \longleftarrow \text{PAM signal}$$

A_k: indep r.v

p: Pulse of arbitrary shape.

$$\therefore \quad E\{x(t)\} = \sum_{k=-\infty}^{\infty} \overset{a}{\cancel{E\{A_k\}}} \, p(t - kT - \Delta)$$

$$= a \sum_{k=-\infty}^{\infty} p(t - kT - \Delta).$$

$$R_x(t+\tau, t) = E\Big\{ \sum_k \sum_j A_k A_j \, p(t - kT - \Delta) \, p(t + \tau - jT - \Delta) \Big\}$$

let $j = k'$

$k = k' + m$

Let $E\{A_k A_{k+m}\} = \alpha_{|m|}$

$$\therefore \quad R_x(t+\tau, t) = \sum_{m=-\infty}^{\infty} \alpha_{|m|} \sum_{k'=-\infty}^{\infty} p(t - (k'+m)T - \Delta) \, p(t + \tau - k'T - \Delta)$$

Replacing t by $t+T$ in $E\{x(t)\}$ and $R_x(t+\tau, t)$ shows that this process is Cyclostationary.

Now, let Δ be uniformly distributed in $(-\frac{T}{2}, \frac{T}{2})$.

then

$$E\{X(t)\} = a \int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=-\infty}^{\infty} p(t-kT-\delta) \cancel{p_\Delta(\delta)}^{1/T} \, d\delta$$

$$= \frac{a}{T} \sum_{k=-\infty}^{\infty} \int_{t-[k-\frac{1}{2}]T}^{t-[k+\frac{1}{2}]T} p(u)\, du \quad = \frac{a}{T} \int_{-\infty}^{\infty} p(t)\,dt \quad \Leftarrow \text{indep of } t.$$

$$R_X(t+\tau, t) = \sum_{k,m=-\infty}^{\infty}\sum \alpha_{|m|} \int_{-T/2}^{T/2} p(t+\tau-kT-\delta)\, p(t-(k+m)T-\delta)\, \frac{d\delta}{T}$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \alpha_{|m|} \sum_{k=-\infty}^{\infty} \int_{t-[k-\frac{1}{2}]T}^{t-[k+\frac{1}{2}]T} p(u+\tau)\, p(u-mT)\, du$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \alpha_{|m|} \int_{-\infty}^{\infty} p(t'+\tau+mT)\, p(t')\, dt'$$

$$= \frac{1}{T} \sum_{m=-\infty}^{\infty} \alpha_{|m|}\, r(\tau+mT) \quad \leftarrow \text{indep of } t.$$

where

$$r(\tau) = \int_{-\infty}^{\infty} p(t+\tau)\, p(t)\, dt.$$

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