state estimation:

Filtering (the Kalman Filter)

Know,

$$\hat{x}(k+1|k+1) = E \left\{ x(k+1) | z(k), z(k+1) \right\}$$

$$= E \left\{ x(k+1) | \underbrace{z(1), z(2), \ldots, z(k), z(k+1)}_{Z(k)} \right\}$$

$$= E \left\{ x(k+1) | Z(k), z(k+1) \right\}$$

Note that $Z(k)$ and $z(k+1)$ are dependent,

because

$$Z(k) = \{ z(1), z(2), \ldots, z(k) \} \rightarrow \{ x(1), x(2), \ldots, x(k) \}$$

$$z(k+1) = H(k+1) x(k+1) + V(k+1) \rightarrow x(k)$$

$$\underbrace{\hat{x}(k) + K(z(k) - H(k) \hat{x}(k))}_{\text{KF}}$$

\therefore $Z(k)$ and $z(k+1)$ both depend on $x(k)$ and are, therefore, dependent.
Recall that $X(k+1)$, $Z(k)$, and $\hat{Z}(k)$ are jointly Gaussian random vectors; hence,

$$\hat{X}(k+1 | k+1) = \mathbb{E} \left\{ X(k+1) \/ Z(k), \hat{Z}(k+1 | k) \right\}$$

where

$$\hat{Z}(k+1 | k) = Z(k+1) - \mathbb{E} \left\{ Z(k+1) \/ Z(k) \right\}.$$

Recall if $x$, $y$, and $z$ are jointly Gaussian and if $y$ and $z$ are dependent,

$$\mathbb{E} \{ x | y, z \} = \mathbb{E} \{ x | y, \hat{z} \}$$

where

$$\hat{z} = z - \mathbb{E} \{ z \/ y \}.$$

So that

$$\mathbb{E} \{ x | y, z \} = \mathbb{E} \{ x | y \} \/ \mathbb{E} \{ z \} - \mathbb{E} \{ x \}.$$
\[
\hat{x}(k+1 | k) = E \{ x(k+1) \mid z(k), \tilde{z}(k+1 | k) \} \\
= E \{ x(k+1) \mid z(k) \} + E \{ x(k+1) \mid \tilde{z}(k+1 | k) \} - \mathbf{m}_x(k+1)
\]

We will come back to this equation later.

It can be shown that

\[
\hat{x}(k+1 | k+1) = \hat{x}(k+1 | k) + K(k+1) \tilde{z}(k+1 | k).
\]

\[\text{Kalman gain}\]

\[\text{Predictor}\]

\[\text{Correction}\]

Where \(K(k+1)\) is yet to be determined.

Remark: Predictor uses information from the state \(\tilde{z}\), because

\[
\hat{x}(k+1 | k) = \Phi(k+1, k) \hat{x}(k+1 | k) + \varpi(k+1) \quad \text{where } \varpi(k) = \text{new measurement available at } k+1.
The error covariance matrix \( P(k+1|k+1) \) for the Kalman filter

\[
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K(k+1) \tilde{z}(k+1|k) \tag{1}
\]

is

\[
P(k+1|k+1) = [I - K(k+1)H(k+1)]P(k+1|k)[I - K(k+1)H(k+1)]^T + K(k+1)R(k+1)K(k+1)^T.
\]

**Proof.** Recall

\[
\tilde{z}(k+1|k) = H(k+1)\tilde{x}(k+1|k) + V(k+1) \tag{2}
\]

(2) in (1) \( \Rightarrow \)

\[
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K(k+1) \left[ H(k+1)\tilde{x}(k+1|k) + V(k+1) \right] \tag{3}
\]

Subtract (3) from \( x(k+1) \)

\[
\tilde{x}(k+1|k+1) = \tilde{x}(k+1|k) - K(k+1) \left[ H(k+1)\tilde{x}(k+1|k) + V(k+1) \right]
\]

\[
\therefore \tilde{x}(k+1|k+1) = \left[ I - K(k+1)H(k+1) \right] \tilde{x}(k+1|k) - K(k+1)V(k+1) \tag{4}
\]

Now \( P(k+1|k+1) = \mathbb{E}\left\{ \tilde{x}(k+1|k+1)\tilde{x}^T(k+1|k+1) \right\} \)

\[
(5)
\]
\[ P(k+1|k+1) = E \left\{ \left( \begin{bmatrix} I - K^{(k+1)} H^{(k+1)} \end{bmatrix} \tilde{x}(k+1|k) - K^{(k+1)} V^{(k+1)} \right) \right\} \]

\[ = \left[ I - K^{(k+1)} H^{(k+1)} \right] P^{(k+1|k)} \left[ I - K^{(k+1)} H^{(k+1)} \right]^\prime + K^{(k+1)} R^{(k+1)} K^{(k+1)} \]  

Notice that \( \tilde{x}(k+1|k) \) and \( V^{(k+1)} \) are indep and therefore the cross-multiplication is zero above.

Recall

\[ P(k+1|k) = \Phi^{(k+1, k)} P(k|k) \Phi^{(k+1, k)} + \Gamma^{(k+1|k)} Q(k) \Gamma^{(k+1|k)} \]  

\[ \left( \Phi = \frac{16}{11} \right) \]  

\[ \left( \frac{16}{11} \right) \]
Remark 1.

Notice that (6) and (7) can be computed recursively, once \( K(k+1) \) is specified, as follows:

\[
P(0 \mid 0) \rightarrow P(1 \mid 0) \rightarrow P(1 \mid 1) \rightarrow P(2 \mid 1) \rightarrow P(2 \mid 2) \rightarrow \cdots
\]

- \( P(0 \mid 0) \) is obtained from (7).
- \( P(1 \mid 0) \) obtained from (6).
- \( P(1 \mid 1) \) obtained from (8).
- \( P(2 \mid 1) \) obtained from (9).
- \( P(2 \mid 2) \) obtained from (6).

Remark 2.

Eq. (6) is true for any gain matrix, including the optimal gain matrix obtained later.
a. The mean-squared filtered estimator of $\hat{x}(k+1)$, $\hat{x}(k+1|k+1)$, written in predictor-corrector format, is

$$\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K(k+1) \tilde{z}(k+1|k)$$

where $\hat{x}(0|0) = m_x(0)$, and $\tilde{z}(k+1|k)$ is the innovations process.

b. $K(k+1)$ is an $n \times m$ matrix which is specified as follows:

$$K(k+1) = P(k+1|k) H'(k+1) \left[ H(k+1) P(k+1|k) H'(k+1) + R(k+1) \right]^{-1}$$

and

$$P(k+1|k+1) = (I - K(k+1) H(k+1)) P(k+1|k)$$

where

$$\tilde{z}(k+1|k+1) = x(k+1) - \hat{x}(k+1|k+1)$$

is a zero-mean Gaussian Markov sequence whose covariance matrix is given in (b).
pf.

a. Recall

\[
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + E\left\{ x(k+1) \mid \hat{z}(k+1|k) \right\} - m_x(k+1)
\]

know \( \hat{x}(k+1) \) and \( \hat{z}(k+1) \) are jointly Gaussian.

Because \( \hat{z}(k+1) \) and \( \hat{\hat{z}}(k+1) \) are causally invertible, i.e.,

we can compute one from the other using a causal system, \( x(k+1) \) and \( \hat{\hat{z}}(k+1|k) \) are also jointly Gaussian.

\[
E\left\{ x(k+1) \mid \hat{\hat{z}}(k+1|k) \right\} = m_x(k+1) + P_{x\hat{z}}(k+1, k+1|k) P_{\hat{\hat{z}}\hat{z}}^{-1}(k+1|k)\left[ m_{\hat{\hat{z}}}(k+1|k) - \hat{m}_{\hat{\hat{z}}}(k+1|k) \right]
\]

Note that \( m_{\hat{\hat{z}}} = 0 \).

Let

\[
K(k+1) = P_{x\hat{z}}(k+1, k+1|k) P_{\hat{\hat{z}}\hat{z}}^{-1}(k+1|k)
\]

\[
\therefore (2), (3) \text{ in (1)} = 0
\]

\[
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K(k+1) \hat{z}(k+1|k)
\]

\( k = 0, 1, 2, \ldots \).
b. \[ P_{x^2} = E \left\{ \left[ x(k+1) - m_x(k+1) \right] \left[ \tilde{x}(k+1) - m_{\tilde{x}(k+1)} \right] \right\} \]

\[ = E \left\{ \left[ x(k+1) \tilde{x}'(k+1) \right] \right\} - m_x(k+1) E \left\{ \tilde{x}(k+1) \right\} \]

Recall

\[ \tilde{x}(k+1) = H(k+1) \tilde{x}(k+1|k) + v(k+1) \]  \hspace{1cm} (2)

\[ (2) \text{ in } (1) \rightarrow 0 \]

\[ P_{x^2} = E \left\{ x(k+1) \left[ H(k+1) \tilde{x}(k+1|k) + v(k+1) \right] \right\} \]

\[ = E \left\{ x(k+1) \tilde{x}'(k+1|k) \right\} H'(k+1) + E \left\{ x(k+1)^2 \right\} E \left\{ v(k+1) \right\} \]  \hspace{1cm} (3)

\[ \text{Known} \]

\[ x(k+1) = \tilde{x}(k+1|k) + \hat{x}(k+1|k) \]  \hspace{1cm} (4)

\[ (4) \text{ in } (3) \rightarrow 0 \]

\[ P_{x^2} = E \left\{ \left[ \tilde{x}(k+1|k) + \hat{x}(k+1|k) \right] \tilde{x}'(k+1|k) \right\} H'(k+1) \]

\[ = E \left\{ \tilde{x}(k+1|k) \tilde{x}'(k+1|k) \right\} H'(k+1) + E \left\{ \hat{x}(k+1|k) \tilde{x}'(k+1|k) \right\} H'(k+1) \]

\[ = \underbrace{P(k+1|k)}_{P(\hat{x}(k+1|k))} \text{ Orthogonality principle} \]

\[ : P_{x^2} = P(k+1|k) H'(k+1). \]  \hspace{1cm} (5)
Also recall that the covariance matrix of the innovations is

$$P_{\tilde{z}}(k+1|k) = H(k+1)P(k+1|k)H'(k+1) + R(k+1) \quad (6)$$

$$K^{-1}(k+1) = P_{\tilde{z}}^{-1}(k+1, k+1|k)P_{\tilde{z}}(k+1|k)$$

$$= P(k+1|k)H'(k+1) \left[ H(k+1)P(k+1|k)H'(k+1) + R(k+1) \right]^{-1} \quad (7)$$

The state prediction error covariance matrix $P(k+1|k)$ was derived in lesson 16.

The state filtering error covariance matrix $P(k+1|k+1)$ is obtained as follows:

Known:

$$P(k+1|k+1) = \left[ I - KH \right]P \left[ I - KH \right]' + KRK' \quad (8)$$
\[ P(k+1|k+1) = (I-kH)P I' - (I-kH)PH'k' + kRk' \]
\[ = (I-kH)P - PH'k' + kHPh'k' + kRk' \]
\[ = k (HPh' + R)k' \]

Known from (7)

\[ K = PH' \left( HPH' + R \right)^{-1} = 0 \quad K(HPh' + R) = PH' \] \hspace{1cm} (10)

(10) in (9) \Rightarrow 0

\[ P(k+1|k+1) = (I-kH)P - PH'k' + PH'k' = (I-kH)P \]

\[ \therefore P(k+1|k+1) = \left[ I - K(k+1)H(k+1) \right]P(k+1|k) \]

C. The Pf of \( \tilde{x}(k+1|k+1) \) is 2.n. Gaussian, Markov is simple. (see next Pg).