The actual state error for $r = 1$
The actual state error for $r = 100$
\[ x(k+1) = x(k) + w(k) \]
\[ z(k+1) = x(k+1) + v(k+1) \]

\[ P(0) = 100 \]
\[ q = 25 \]
\[ r = 15 \]

**Note:** \( \Phi = 1, \quad H = 1 \)

\[ P(k+1|k) = \frac{P(k|k) + q}{P(k|k) + q} \]
\[ K(k+1) = \frac{P(k|k)}{P(k|k) + r} \]
\[ P(k+1|k+1) = \left(1 - K(k+1)\right) P(k+1|k) \]
\[ \hat{x}(k+1|k) = \hat{x}(k|k) \]
\[ \tilde{z}(k+1|k) = z(k+1) - \hat{x}(k+1|k) \]

\[ \hat{x}(k+1|k+1) = \hat{x}(k|k) + K(k+1) \tilde{z}(k+1|k) \]
Estimate the value of a constant $\bar{x}$, given discrete measurements of $x$ corrupted by an uncorrelated Gaussian noise sequence with variance $\sigma^2$.

**Solution:**

\[ x(k+1) = x(k) \]
\[ z(k+1) = x(k+1) + v(k+1) \]

\[
\begin{align*}
P(k+1|k) &= \Phi P(k|k) \Phi' + Q \Gamma \\
K(k+1) &= P(k+1|k) H' [ H' P(k+1|k) H + R(k) ]^{-1} \\
P(k+1|k+1) &= [ I - K(k+1) H ] P(k+1|k)
\end{align*}
\]

\[
P(k+1|k) = P(k|k) \quad (1)
\]
\[
K(k+1) = \frac{P(k+1|k)}{P(k+1|k) + \sigma^2} \quad (2)
\]
\[
P(k+1|k+1) = (1 - K(k+1)) P(k+1|k) \quad (3)
\]

\[
(1) \text{ in } (2) \Rightarrow K(k+1) = \frac{P(k|k)}{P(k|k) + \sigma^2} \quad (4)
\]
\[
(4) \text{ in } (3) \Rightarrow P(k+1|k+1) = \left[ 1 - \frac{P(k|k)}{P(k|k) + \sigma^2} \right] P(k|k)
\]
\[ P(k+1/k+1) = \frac{r}{P(k/k)+r} \quad P(k/k) = \frac{P(k/k)}{1 + \frac{P(k/k)}{r}} \quad (5) \]

Therefore, \( P(k+1/k+1) \) can be obtained by solving the nonlinear difference Eq. (5) as follows:

Let \( P(0/0) = P_x(0) \)

\[ (5) = 0: \quad P(1/1) = \frac{P_x(0)}{1 + \frac{P_x(0)}{r}} , \quad P(2/2) = \frac{P_x(0)}{1 + \frac{2P_x(0)}{r}} , \]

\[ \ldots \ldots \ldots \ldots \ldots \quad P(k/k) = \frac{P_x(0)}{1 + \frac{kP_x(0)}{r}} \quad (6) \]

(6) in (4) = 0

\[ K_x(k+1) = \frac{P_x(0) / 1 + \frac{kP_x(0)}{r}}{1 + \frac{kP_x(0)}{r}} + r = \frac{P_x(0)}{P_x(0) + r (1 + \frac{kP_x(0)}{r})} \]

\[ = \frac{P_x(0) / r}{1 + (k+1) \frac{P_x(0)}{r}} \quad (7) \]
Also, know:
\[
\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + K \hat{z}(k+1|k) + \hat{z}(k+1|k) - \hat{x}(k+1|k)
\]

\[
\hat{x}(k+1|k+1) = \hat{x}(k|k) + K \hat{z}(k+1|k) - \hat{x}(k|k)
\]

\[
\begin{align*}
\hat{x}(k+1|k+1) &= \hat{x}(k|k) + \frac{P_{x}(o)/r}{1 + (k+1) P_{x}(o)/r} \left[ \hat{z}(k+1) - \hat{x}(k|k) \right] \\
\hat{x}(0|0) &= m_{x}(0)
\end{align*}
\]

Remark. Consider the problem of estimating a scalar non-random constant \( x \) based on \( k \) noisy measurements, \( z_{i} \), where

\[
z_{i} = x + v_{i} \quad (i = 1, 2, \ldots, k)
\]
\[ \hat{X}(k|k) = \frac{1}{k} \sum_{i=1}^{k} Z_i \]

\[ \hat{X}(k+1|k+1) = \frac{1}{k+1} \sum_{i=1}^{k+1} Z_i = \frac{1}{k+1} \sum_{i=1}^{k} Z_i + \frac{1}{k+1} Z_{k+1} \]

\[ = \frac{K}{k+1} \left( \frac{1}{k} \sum_{i=1}^{k} Z_i \right) + \frac{1}{k+1} Z_{k+1} \]

\[ = \frac{K+1}{k+1} \hat{X}(k|k) + \frac{1}{k+1} Z_{k+1} \]

or

\[ \hat{X}(k+1|k+1) = \hat{X}(k|k) + \frac{1}{k+1} (Z_{k+1} - \hat{X}(k|k)) \]

Notice that this result is the same as the Kalman filter by letting \( P_x(0) \to \infty \).
% kal
% M file to simulate the Kalman filter

hold off
clear
clc
r = 10; % variance of the measurement noise
x(1) = 4; % initial value of the state
p(1) = 0.1; % variance of the initial state
xhf(1) = 0; % filter estimate of the initial state
N = 10000; % number of samples
fi = 1;

% Dynamic Equations

for i = 1:N
    x(i+1) = fi*x(i);
end
n = [1:N+1];
z = x + randn(size(n))*sqrt(r);

% Error Variance and Kalman Gain

for i = 1:N
    k(i+1) = p(i)/(p(i) + r);
    p(i+1) = r*p(i)/(p(i) + r);
end

% Kalman Estimator

for i = 1:N
    xhp(i+1) = fi*xhf(i);
    ztil(i+1) = z(i+1) - xhp(i+1);
    xhf(i+1) = xhp(i+1) + k(i+1)*ztil(i+1);
    error(i+1) = x(i+1) - xhf(i+1);
end

% Outputs

plot(x)
hold
plot(xhf,'r')
hold off
label = sprintf('Actual State');
title (label)
ylabel ('State - x')
xlabel ('Time')
pause
plot (z)
label = sprintf ('The Measurements for r = %g', r);
title (label)
ylabel ('z')
xlabel ('Time')
pause
plot (k(2:length(k)))
label = sprintf ('Kalman Gain for r = %g', r);
title (label)
ylabel ('K')
xlabel ('Time')
pause
plot (p)
label = sprintf ('State Error Variance for r = %g', r);
title (label)
ylabel ('State Error Variance')
xlabel ('Time')
pause
plot (error)
label = sprintf ('Actual Error for r = %g', r);
title (label)
ylabel ('Actual Error')
xlabel ('Time');
pause