Chapter 4

Image Sampling and Quantization

The most basic requirement for computer processing of images is that the images be given in digital form, that is, as arrays of finite length binary words. For digitization, the given image is sampled on a discrete grid and each sample (or pixel) is quantized using a finite number of bits. The digitized image can then be processed by the computer. To display a digital image, it is first converted to an analog signal, which is scanned onto a display.

\[ f(x, y) \rightarrow \text{Sampler} \rightarrow f_s(x, y) \rightarrow \text{Quant.} \rightarrow u^*(m, n) \rightarrow \text{Digital Computer} \]

Digitization process.

\[ \text{Digital Computer} \rightarrow u^*(m, n) \rightarrow \text{D/A} \rightarrow \text{Display} \]

Display process.
As of

\[ C_{\mathrm{om}}(t) = \sum_{n} \delta(t - nT) \]

\[ F\{C_{\mathrm{om}}(t)\} = ? \]

\[ C_{\mathrm{om}}(t) = \text{periodic with period } T \]

\[ \therefore C_{\mathrm{om}}(t) = \sum_{m} a_{m} e^{jmw_{0}t} \quad , \quad w_{0} = \frac{2\pi}{T} \]

\[ \therefore F\{C_{\mathrm{om}}(t)\} = \mathcal{F}\left\{ \sum_{m} a_{m} e^{jnw_{0}t} \right\} \]

\[ = \sum_{m} a_{m} F\{e^{jnw_{0}t}\} \]

\[ = \sum_{m} a_{m} \delta(w - mw_{0}) \]

But \[ a_{m} = \frac{1}{T} \int_{0}^{T} C_{\mathrm{om}}(t) \, dt = \frac{1}{T} \]

\[ \therefore F\{C_{\mathrm{om}}(t)\} = \sum_{m} \frac{1}{T} \, \delta(w - mw_{0}) = \frac{w_{0}}{T} \sum_{m} \delta(w - mw_{0}) \]

\[ \overbrace{C_{\mathrm{omB}1}(w)}^{\text{Fourier Transform of Comb is Comb!}} \]
TWO-DIMENSIONAL SAMPLING THEORY

Def - A function \( f(x,y) \) is said to be bandlimited if its Fourier transform satisfy the following:

\[
F(\xi_1, \xi_2) = 0 \quad \quad |\xi_1| > \xi_{x_0}, \quad |\xi_2| > \xi_{y_0}
\]

The quantities \( \xi_{x_0} \) and \( \xi_{y_0} \) are called the x and y bandwidths of the image.

\[\text{SAMPLING THEOREM:}\]

The Fourier transform of an arbitrary sampled function is a scaled, periodic replication of the Fourier transform of the original function.

Proof:

Consider the ideal sampling function

\[
\text{comb}(x,y; \Delta x, \Delta y) = \sum_{m} \sum_{n} \delta(x - m \Delta x, y - n \Delta y)
\]
The sampled image is defined as

\[ f_s(x, y) = f(x, y) \text{comb}(x, y; \Delta x, \Delta y) \]
\[ = \sum_{m} \sum_{n} f(m \Delta x, n \Delta y) \delta(x - m \Delta x, y - n \Delta y) \]

Taking the Fourier transform of both sides yields

\[ \frac{1}{\sqrt{2\pi}} F_s(\xi_1, \xi_2) = F(\xi_1, \xi_2) \ast \text{COMB}(\xi_1, \xi_2) \]

But, the Fourier transform of a comb function with spacing ($\Delta x$, $\Delta y$) is another comb function with spacing (1/$\Delta x$, 1/$\Delta y$), namely,

\[ \text{COMB}(\xi_1, \xi_2) = (\lambda/\Delta x, \lambda/\Delta y) \text{comb}(\xi_1, \xi_2; 1/\Delta x, 1/\Delta y) \]
\[ = (\lambda/\Delta x, \lambda/\Delta y) \sum_{k} \sum_{l} \delta[\xi_1 - k (1/\Delta x), \xi_2 - l (1/\Delta y)] \]

Thus,

\[ \frac{1}{\sqrt{2\pi}} F_s(\xi_1, \xi_2) = F(\xi_1, \xi_2) \ast \text{COMB}(\xi_1, \xi_2) \]
\[ = F(\xi_1, \xi_2) \ast \left(1/\Delta x\right) \left(1/\Delta y\right) \sum_{k} \sum_{l} \delta[\xi_1 - k (1/\Delta x), \xi_2 - l (1/\Delta y)] \]
\[ = \left(1/\Delta x\right) \left(1/\Delta y\right) \sum_{k} \sum_{l} F(\xi_1, \xi_2) \ast \delta[\xi_1 - k (1/\Delta x), \xi_2 - l (1/\Delta y)] \]
\[ = \left(1/\Delta x\right) \left(1/\Delta y\right) \sum_{k} \sum_{l} F(\xi_1 - k (1/\Delta x), \xi_2 - l (1/\Delta y)] \]

which shows that the Fourier transform of the sampled image is, within a scale factor, a periodic replication of the Fourier transform of the input image on a grid whose spacing is (1/$\Delta x$, 1/$\Delta y$).
(b) Sampled image spectrum.

(c) Aliasing and foldover frequencies (shaded areas).

Figure 4.7 Two-dimensional sampling.
RECONSTRUCTION OF THE IMAGE FROM ITS SAMPLES

From the previous figure it is seen that if

\[ \xi_{xs} > 2\xi_{x0}, \quad \xi_{ys} > 2\xi_{y0} \]

or equivalently, if the sampling intervals are such that

\[ \Delta x < 1/2\xi_{x0}, \quad \Delta y < 1/2\xi_{y0} \]

Then \( F(\xi_1, \xi_2) \) can be recovered by a LPF with frequency response

\[
H(\xi_1, \xi_2) = \begin{cases} 
\frac{1}{\xi_{xs} \xi_{ys}} & (\xi_1, \xi_2) \in \mathbb{R} \\
0 & \text{otherwise}
\end{cases}
\]

In other words,

\[
\tilde{F}(\xi_1, \xi_2) = H(\xi_1, \xi_2) F_s(\xi_1, \xi_2) = F(\xi_1, \xi_2)
\]

which shows that the original signal can be recovered exactly by low-pass filtering the sampled image.

Remarks

1. The lower bounds on the sampling rates, that is, \( 2\xi_{x0} \) and \( 2\xi_{y0} \), are called the Nyquist rates or the Nyquist frequencies.

2. If the sampling frequencies are below the Nyquist frequencies, that is;

\[ \xi_{xs} < 2\xi_{x0}, \quad \xi_{ys} < 2\xi_{y0} \]

then the periodic replications of \( F(\xi_1, \xi_2) \) will overlap, resulting in a distorted spectrum \( F_s(\xi_1, \xi_2) \), from which \( F(\xi_1, \xi_2) \) is lost.
3. The frequencies above half the sampling frequencies, that is, above $\xi_{xs}/2$, $\xi_{ys}/2$, are called the fold-over frequencies.

**Sampling Theorem (p. 88 text):**

A bandlimited image $f(x,y)$ satisfying

$$F(\xi_1, \xi_2) = 0, \quad |\xi_1| > \xi_{x0}, \quad |\xi_2| > \xi_{y0}$$

and sampled uniformly on a rectangular grid with spacing $\Delta x$, $\Delta y$ can be recovered without error from the sample values $f(m\Delta x, n\Delta y)$ provided the sampling rate is greater than the Nyquist rate, that is,

$$\xi_{xs} > 2\xi_{x0}, \quad \xi_{ys} > 2\xi_{y0}$$

Moreover, the reconstructed image is given by the interpolation formula

$$f(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y) \left( \frac{\sin \left( \pi \frac{x-x_m}{\Delta x} \right) \pi}{\left( \pi \frac{x-x_m}{\Delta x} \right) \pi} \right) \left( \frac{\sin \left( \pi \frac{y-y_n}{\Delta y} \right) \pi}{\left( \pi \frac{y-y_n}{\Delta y} \right) \pi} \right)$$
Exn
\[ f(x, y) = 2 e^{j2\pi (3x + 4y)} \]

\[ \Delta x = \Delta y = 0.2 \quad \Rightarrow \quad \int_{x_1}^{x_2} - \int_{y_1}^{y_2} = \frac{1}{0.2} = 5 \]

\[ f(x, y) = \frac{e^{j2\pi (3x + 4y)}}{x} + \frac{e^{-j2\pi (3x + 4y)}}{x} \]

\[ \therefore F(s_1, s_1) = \delta(s_1 - 3, s_1 - 4) + \delta(s_1 + 3, s_1 + 4) \quad (*) \]

Note that:
\[ (s_{x_1}, s_{y_3}) < 2 (s_{x_0}, s_{y_0}) \]
\[ (5, 5) < (6, 8) \]
\((x) = 0 \quad F_3(f_1, f_2) = \sum_{\kappa, \ell = -\infty}^{\infty} \delta(f_1 - 3 - 5\kappa, f_2 - 4 - 5\ell) + \delta(f_1 + 3 - 5\kappa, f_2 + 4 - 5\ell)\)

Let 
\[ H(f_1, f_2) = \begin{cases} \frac{1}{c_5} & (f_1, f_2) \in \mathbb{R} \\ 0 & \text{elsewhere} \end{cases} \]

\[ F_3(f_1, f_2) H(f_1, f_2) = \delta(f_1 - 2, f_2 - 1) + \delta(f_1 + 2, f_2 + 1) \]

\[ \hat{f}(x, y) = 2c_0 \pi(2x + y) \]
In physical sampling environments, random noise is always present in the image, so it is important to consider sampling theory for random fields.

**Def** A finite stationary random field \( f(x, y) \) is called band limited if

\[
S(f, f_0) = 0 \quad \text{for} \quad \|f\| > f_{x_0}, \quad \|f\| > f_{y_0}
\]

**Sampling theorem for random fields (P. 90)**

If \( f(x, y) \) is a stationary band limited random field, then

\[
\tilde{f}(x, y) = \sum_{m, n = -\infty}^{\infty} f(m, n) \Delta x \Delta y \sin\left(\frac{2\pi x}{x_s} - m\right) \sin\left(\frac{2\pi y}{y_s} - n\right)
\]

Converges to \( f(x, y) \) in the MSE sense, i.e.,

\[
E\|f(x, y) - \tilde{f}(x, y)\|^2 = 0
\]

where \( f_{x_0} = \frac{1}{\Delta x}, \quad f_{y_0} = \frac{1}{\Delta y}, \quad f_{x_s} > 2f_{x_0}, \quad f_{y_s} > 2f_{y_0} \).
1. If the random field \( f(x,y) \) is sampled above its Nyquist rate, then a continuous random field \( \tilde{f}(x,y) \) can be obtained (reconstructed) from the sampled sequence such that \( \tilde{f}(x,y) \) converges to \( f(x,y) \) in MSE sense.

2. It can be shown that when

\[
S_s(f_1, f_2) = \text{PSD} \{ \tilde{f}(x,y) \}
\]

\[
S_s(f_1, f_2) = \text{PSD} \{ f(x,y) \}
\]

Then,

\[
S_s(f_1, f_2) \approx \sum_{x,y} \sum_{k,p} S(h_1 - k, h_2 - l) S(x, y)
\]

That is, the PSD of the sampled image is a periodic extension of the PSD of \( f(x,y) \).
**Image Quantization**

The step after the sampling in image digitization is quantization. A quantizer maps a continuous variable $u$ into a discrete variable $u'$, which takes values from a finite set $(r_1, r_2, ..., r_L)$ of numbers. This mapping is generally a staircase function and the rule is:

Define $(t_k, k = 1, ..., L+1)$ as a set of increasing transition or decision levels with $t_1$ and $t_{L+1}$ as the minimum and maximum values, respectively, of $u$. If $u$ lies in the interval $[t_k, t_{k+1})$, then it is mapped to $r_k$, the $k$th reconstruction level.

![Diagram of a quantizer](image)

*Figure 4.16* A quantizer.
Sampling and digital conversion process.
THE OPTIMUM MEAN-SQUARE OR LLOYD-MAX QUANTIZER

This quantizer minimizes the MSE for a given number of quantization levels. Let \( u \) be an rv with continuous pdf \( p(u) \). It is desired to find the decision levels \( t_k \) and the reconstruction levels \( r_k \) for an \( L \)-level quantizer such that

\[
MSE = E[(u - u')^2] = \int_{t_1}^{t_{L+1}} (u - u')^2 p(u) \, du \quad (*)
\]

is minimized.

\[(*) \implies MSE = \int_{t_1}^{t_2} (\quad) + \int_{t_2}^{t_3} (\quad) + \cdots + \int_{t_L}^{t_{L+1}} (\quad)
\]

\[= \sum_{i=1}^{L} \int_{t_i}^{t_{i+1}} \frac{r_i}{r_i} (u - u')^2 p(u) \, du
\]

\[\therefore \frac{\partial MSE}{\partial t_k} = 0 \quad \text{and} \quad \frac{\partial MSE}{\partial r_k} = 0
\]

Together, these constraints lead to the following simultaneous equations:

\[t_k = \frac{r_k + r_{k-1}}{2} \quad ; \quad k = 1, 2, \ldots, L.
\]

\[r_k = \frac{\int_{t_k}^{t_{k+1}} u \, p(u) \, du}{\int_{t_k}^{t_{k+1}} p(u) \, du}
\]
\[ P(u) = \begin{cases} \frac{1}{t_{L+1} - t_1}, & t_1 \leq u \leq t_{L+1} \\ 0, & \text{otherwise} \end{cases} \]

Recall that

\[ r_k = \frac{\int_{t_k}^{t_{k+1}} u P(u) \, du}{\int_{t_k}^{t_{k+1}} P(u) \, du} = \frac{\frac{u^2}{2} \bigg|_{t_k}^{t_{k+1}}}{t_{k+1} - t_k} = \frac{t_{k+1}^2 - t_k^2}{2(z(t_{k+1} - t_k)}} \]

\[ r_k = \frac{t_{k+1} + t_k}{2} \quad (\# \#) \]

Also,

\[ t_k = \frac{r_k + r_{k-1}}{2} = \frac{t_{k+1} + t_k}{2} + \frac{t_k + t_{k-1}}{2} = \frac{t_{k+1} + t_{k-1} + 2t_k}{4} \]

\[ t_{k+1} + t_{k-1} + 2t_k = 4t_k \]

\[ t_k = \frac{t_{k+1} + t_{k-1}}{2} \quad (\# \#) \]
\[(*) \quad t_{k+1} + t_{k-1} = \frac{2t_k}{t_k + t_k} \]
\[\therefore t_k - t_{k-1} = t_{k+1} - t_k = \text{constant} \quad \Delta q \]

\[ \begin{array}{c}
\text{q} \\

\hline
\hline
\hline

\end{array} \]

\[ t_{L+1} - t_1 = L \cdot q \]
\[ \therefore q = \frac{t_{L+1} - t_1}{L} \]
\[ t_k = t_{k-1} + \frac{q}{L} \]

\[(**): \quad r_k = \frac{t_{k+1} + t_k}{2} \]
\[ = \frac{(t_k + q) + t_k}{2} \]
\[ = t_k + \frac{q}{2} \]
Example:

The most common quantizer is the uniform one. Let the output of an image sensor take values between 0.0 to 10.0. If the samples are quantized uniformly to 256 levels, then the transition and reconstruction levels are

\[ t_k = \frac{10(k-1)}{256}, \quad k = 1, \ldots, 257 \]

\[ r_k = t_k + \frac{5}{256}, \quad k = 1, \ldots, 256 \]

The interval \( q = t_k - t_{k-1} = r_k - r_{k-1} \) is constant for different values of \( k \) and is called the quantization interval.
VISUAL QUANTIZATION

The previous methods can be applied for gray scale quantization of monochrome images. If the number of quantization levels is not sufficient, a phenomenon called **contouring** becomes visible. When groups of neighboring pixels are quantized to the same value, regions of constant gray levels are formed, whose boundaries are called **contours**. Uniform quantization of common images, where the pixels represent the luminance function, requires about 256 gray levels, or 8 bits. Contouring effects start becoming visible at or below 6 bits/pixel.

In evaluating quantized images, the eye seems to be quite sensitive to contours and errors that affect local structure. However, the contours do not contribute very much to the MSE. Thus a visual quantization scheme should attempt to hold the quantization contours below the level of visibility over the range of luminances to be displayed. We consider two methods achieving this (other than allocating the full 8 bits/pixel).
1. **Contrast Quantization**

Since visual sensitivity is nearly uniform to noticeable changes in contrast, it is more appropriate to quantize the contrast function as shown below.

Two nonlinear transformations that have been used for representation of contrast $c$ are

\[ c = \alpha \ln (1 + \beta u) \]
\[ c = \alpha u^\beta \]

where $\alpha$ and $\beta$ are constants and $u$ represents the luminance. For the project we use the second one with $\alpha = 1$ and $\beta = 1/3$.

**Remark**

Experimental studies indicate that a 2% change in contrast is just noticeable. Therefore, if uniformly quantized, the contrast scale needs 50 levels, or about 6 bits. However, with optimum MSE quantizer, 4 to 5 bits/pixel could be sufficient.
2. Pseudorandom Noise Quantization

Another method suppressing contouring effects is to add a small amount of uniformly distributed pseudorandom noise to the luminance samples before quantization. This noise is called ditter. To display the image, the same (or another) noise sequence is subtracted from the quantizer output. The effect is that in the regions of low-luminance gradients (which are the regions of contours), the ditter causes pixels to go above or below the original decision level, thereby breaking the contours. However, the average value of the quantized pixels is about the same with and without the ditter. During display, the noise tends to fill in the regions of contours in such a way that the spatial average is unchanged.