Chapter 10:

AN OVERVIEW OF THE FUNDAMENTALS OF THE COMPUTERIZED TOMOGRAPHY
1. Introduction

The invention of the Computerized Tomography (CT) revolutionized the medical imaging in 1973 [1]. Since then, the diagnostic radiology was solely relied on the simple radiation of the X-rays through the desired organ and observation of the resultant "shadows". The X-ray films, however, were occasionally difficult to interpret due to the superimposition of overlaying objects into the same image, and also the fact that adjacent regions were not clearly distinguished; efforts to circumvent these problems were not successful in many cases [2].

The introduction of CT scanners was a fundamental change that greatly improved the problems associated with the conventional diagnostic radiology. In CT, the outgoing X-rays are digitized and manipulated to create a more complete and useful images. While the patient is stationary, the X-ray source and detectors are moved around the patient to obtain a set of projections in different directions. The full set of projections are used by the computer to reconstruct a digital image of the radiated "slice". Although, the CT was originally developed for medical applications, it soon found its way in other applications such as soil sample analysis, high explosives (HE) characterization [3], etc.

The objective of this report is to prepare a tutorial overview of the fundamentals of the CT and to review some of the well known reconstruction techniques reported in the literature.

The principle theory of the CT lies on the reconstruction of a function from its line integrals which was first solved by Radon in 1917 [4]. To illustrate the problem, consider Figure 1 wherein f(x,y) denote the absorption coefficient of the object at a point (x,y) in a slice at some fixed value z. If we assume that the illumination consists of an infinitely thin parallel beam of X-rays, the intensity of the detected beam is given by the line integral relation [5]:

\[
I = I_0 \exp \left[- \int_L f(x,y) \, du \right]
\]  

(1)

where \(I_0\) is the intensity of the incident beam, L is the path of the ray, and \(u\) is the distance along L. Defining the observed signal as

\[
g = \ln \frac{I_0}{I}
\]

(2)

Then, as illustrated in Figure 2, we obtain the transformation

\[1\] Tomo is the Greek word for cut.
\[ g(s, \theta) = \int_{\mathbb{L}} f(x, y) du \]  \hspace{1cm} (3)

where \((s, \theta)\) represent the coordinates of the X-ray relative to the object. The image reconstruction problem is to determine \(f(x, y)\) from \(g(s, \theta)\) where \((x, y)\) is referred to as the \textit{spatial domain} and \((s, \theta)\) is referred to as the \textit{projection-space domain}.

This report is organized as follows. Section 2 describes the Radon transform and Section 3 is devoted to the various reconstruction techniques. Finally, in Section 4 we present the final remarks of this report.
Figure 1. Radiation of X-rays are recorded and analyzed for reconstruction of $f(x,y)$. 
Figure 2. The process of transformation of $f(x,y)$ to $g(s,\theta)$ by means of X-ray radiation.
2. The Radon Transform

In view of Figure 3, the Radon transformation can be defined as

\[ g(s, \theta) = \int_{l} f(x, y) du \]

\[ = \int \int f(x, y) \delta(x \cos \theta + y \sin \theta - s) dx dy \]  

\[ = Rf = \text{Radon Transform of } f \]  

The operator \( R \) is referred to as the \textit{projection operator}.

In the rotated coordinate system \((s, u)\), shown in Figure 3, we have

\[ \begin{bmatrix} s \\ u \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]  

or

\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} s \\ u \end{bmatrix} \]  

which upon substitution in (4) results in another form of the Radon transform:

\[ g(s, \theta) = \int_{-\infty}^{\infty} f(s \cos \theta - u \sin \theta, s \sin \theta + u \cos \theta) du \]  

The projection function \( g(s, \theta) \) is also called a \textit{ray-sum}, since it represents the sum of \( f(x, y) \) along a ray at a distance \( s \) and an angle \( \theta \).
Several points are worth mentioning. First, the Radon transform maps the spatial domain \((x,y)\) to the domain \((s,\theta)\); each point in the \((s,\theta)\) space corresponds to a line in the space \((x,y)\). Second, the coordinates \((s,\theta)\) are not the polar coordinates of \((x,y)\). In fact, if \((r,\phi)\) are the polar coordinates of \((x,y)\), then

\[
x = r \cos \phi \\
y = r \sin \phi
\]  

(7)

Figure 3. Change of coordinates from \((x,y)\) to \((s,u)\) to obtain a simpler form of the Radon transform.
and

\[ s = x \cos \theta + y \sin \theta = \]

\[ r \cos \phi \cos \theta + r \sin \phi \sin \theta \]

\[ \therefore s = r \cos(\theta - \phi) \]  \hspace{1cm} (8)

For a fixed point \((r, \phi)\), this gives the locus of all points in \((s, \theta)\).
3. Reconstruction of $f(x,y)$ from $g(s,\theta)$

Determination of $f(x,y)$ from $g(s,\theta)$ can be performed by three different approaches. The first method relies on the back-projection technique which performs another transformation on the projections to obtain a suitable function for reconstruction. The second method uses the Projection Theorem to reconstruct the object in the Fourier domain. The third method is based on the inverse Radon transformation which provides the mathematical foundation for the reconstruction of $f(x,y)$ from $g(s,\theta)$. This method leads to several interesting and practical implementations of the object reconstruction.

3.1 The Back-Projection Operator

Recall that the Radon transform is defined as

$$ g(s,\theta) = \int_{-\infty}^{\infty} f(s \cos \theta - u \sin \theta, s \sin \theta + u \cos \theta) du $$

(6)

Associated with (6), we define the back-projection operator $B$ as

$$ b(x,y) = Bg(s,\theta) = \int_{0}^{s} g(x \cos \theta + y \sin \theta, \theta) d\theta $$

(7)

where $b(x,y)$ is called the back-projection of $g(s,\theta)$. Recall that

$$ s = r \cos(\theta - \phi) $$

(8)

Thus (9) in polar domain becomes

$$ b(x, y) = \int_{0}^{\pi} g(r \cos(\theta - \phi), \theta) d\theta $$

(10)

It is seen from (9) that the back-projection operator $B$ maps a function of $(s,\theta)$ into a function of $(x,y)$ or $(r,\phi)$. In addition, Eqn (10) reveals that the function $b(x,y)$ at any pixel $(x,y)$ requires projections from all directions. Finally, it is noted that
\[ b = Bg = BRf \]  

(11)

It can be shown that [5] \( BRf \) is an image of \( f(x,y) \) blurred by the point-spread function (PSF) \( h(x,y) \) where

\[ h(x,y) = (x^2 + y^2)^{-1/2} \]  

(12)

which results in the spatial transfer function

\[ H(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{-1/2} \]  

(13)

Consequently, as depicted in Figure 4, the object \( f(x,y) \) can be restored from \( b(x,y) \) by an inverse filter with the spatial transfer function

\[ H_r(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{1/2} \]  

(14)
Figure 4. The back-projection function $b(x,y)$ is the result of the transformation on $g(s,\theta)$ which can be related to the object $f(x,y)$ by the inverse filter (14).
3.2 The Fourier-Transform method

There exists an interesting relationship between the two-dimensional Fourier transform of a function and the one-dimensional Fourier transform of its corresponding Radon transform. The following theorem, referred to as the Projection Theorem, summarizes this relationship.

- **The Projection Theorem**

  If

  \[ F(\xi_1, \xi_2) = \mathcal{F}\{f(x, y)\} \]
  \[ G(\xi, \theta) = \mathcal{F}\{g(s, \theta)\} \]

  Then

  \[ G(\xi, \theta) = F(\xi \cos \theta, \xi \sin \theta) \]

  That is, the one-dimensional Fourier transform with respect to \( s \) of the projection \( g(s, \theta) \) is equal to the central slice, at angle \( \theta \), of the two-dimensional Fourier transform of \( f(x, y) \).

  **Proof:**

  We know that

  \[ G(\xi, \theta) = \mathcal{F}\{g(s, \theta)\} = \int_{-\infty}^{\infty} g(s, \theta)e^{-2\pi i s \xi}ds \]  \hfill (16)

  Using (6) in (16), we have

  \[ G(\xi, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s \cos \theta - u \sin \theta, s \sin \theta + u \cos \theta)e^{-2\pi i s \xi}dsdu \]  \hfill (17)

  which after the coordinate transformation of (5a) completes the proof of the theorem:
\[ G(\xi, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \exp\{-j2\pi [x\xi \cos \theta + y\xi \sin \theta]\} \, dx \, dy = F(\xi \cos \theta, \xi \sin \theta) \] (18)

The implication of the projection theorem is very interesting. The theorem simply suggests that we can reconstruct the Fourier transform of the object \( f(x, y) \) from the one-dimensional Fourier transform of the projections \( g(s, \theta) \) for every \( \theta \). Then, \( f(x, y) \) can be obtained from

\[ f(x, y) = \mathcal{F}_2^{-1}\{F(\xi_1, \xi_2)\} \] (19)

This process is shown in Figure 5.
Figure 5. The two-dimensional Fourier transform of the object $f(x,y)$ can be reconstructed from the one-dimensional Fourier transform of the ray sum $g(s,\theta)$.
3.3. The Inverse Radon Transform

It can be shown that \([5]\) \(f(x,y)\) can be retrieved from \(g(s,\theta)\) by the inverse Radon transform as follows:

\[
f(x,y) = \frac{1}{2\pi^2} \int \int_{0}^{\infty} \left( \frac{\partial^2}{\partial s^2} \right) \left[ \frac{x \cos \theta + y \sin \theta - s}{s^2} \right] ds d\theta
\]

(20)

Proof:

We know that, by definition:

\[
f(x,y) = \int \int_{-\infty}^{\infty} F(\xi_1, \xi_2) \exp\{j2\pi[\xi_1 x + \xi_2 y]\} d\xi_1 d\xi_2
\]

(21)

In polar domain we have:

\[
\xi_1 = \xi \cos \theta
\]

\[
\xi_2 = \xi \sin \theta
\]

Therefore:

\[
f(x,y) = \int \int_{0}^{2\pi} F(\xi \cos \theta, \xi \sin \theta) \exp\{j2\pi[\xi \cos \theta + \xi \sin \theta]\} \xi d\xi d\theta
\]

(22)

It can be shown that (22) can be written as:

\[
f(x,y) = \int_{0}^{\pi} [\int_{-\infty}^{\infty} G(\xi, \theta) \exp\{j2\pi[\xi \cos \theta + \xi \sin \theta]\} d\xi] d\theta
\]

(23)

\[
= \int_{0}^{\pi} g(x \cos \theta + y \sin \theta, \theta) d\theta
\]
where

\[ \hat{g}(s, \theta) = \int_{-\infty}^{\infty} |\zeta|G(\zeta, \theta) \exp\{j2\pi \zeta s\} d\zeta \]  

(24)

We now write

\[ |\zeta|G(\zeta, \theta) = \zeta G(\zeta, \theta) \text{sgn}(\zeta) \]  

(25)

Then using (25) along with the convolution theorem in (24) yields

\[ \hat{g}(s, \theta) = \int_{-\infty}^{\infty} \{\zeta G(\zeta, \theta)\} \{\text{sgn}(\zeta)\} e^{j2\pi \zeta s} d\zeta \]

\[ = [\mathcal{F}^{-1}\{\zeta G(\zeta, \theta)\}] \otimes [\mathcal{F}^{-1}\{\text{sgn}(\zeta)\}] \]

\[ = \left[ \frac{1}{j2\pi} \frac{\partial \hat{g}(s, \theta)}{\partial s} \right] \otimes \left[ \frac{-1}{j\pi s} \right] \]

\[ = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{\partial \hat{g}(t, \theta)}{\partial t} \frac{1}{s-t} dt \]  

(26)

which upon substitution in (23) renders

\[ f(x, y) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial \hat{g}(t, \theta)}{\partial t} \frac{1}{s-t} dt d\theta \]  

(27)

This completes the proof of (20).

Several interesting implementation of the Radon inversion can be obtained as a result of the proof of the theorem. In the following we explain these results.

- It is seen from (23) that \( f(x, y) \) is obtained by the back-projection of \( \hat{g} \). Also, it follows from (24) that \( \hat{g} \) is the output of a one-dimensional filter whose frequency-response is \(|\zeta|\). Figure (6a) illustrates this process.
• It is noted from (26) that

$$\hat{g} = \frac{1}{2\pi} HD g$$  \hspace{1cm} (28)

where $H$ is the Hilbert transform defined as

$$H\phi = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(t) \frac{dt}{s-t}$$  \hspace{1cm} (29)

and $D$ is the differentiation operator characterized as

$$D\phi = \frac{\partial \phi(s)}{\partial s}$$  \hspace{1cm} (30)

As a result, the Radon inversion can be implemented as depicted in Figure (6b). This form of the Radon inversion is called the convolution back-projection method.

• Finally, it is noticed from (24) that

$$\hat{g} = \int_{-\infty}^{\infty} G(\zeta, \theta) e^{j2\pi \xi \zeta} d\zeta = Hg$$  \hspace{1cm} (31)

where $H_1$ is a one-dimensional filter whose frequency-response is $|\xi|$. Therefore,

$$\hat{g} = \mathcal{F}_1^{-1}\{|\xi|G(\xi, \theta)\} = \mathcal{F}_1^{-1}\{|\xi|\mathcal{F}_1 g\}$$  \hspace{1cm} (32)

Then, it follows from (23) that

$$f(x, y) = B\hat{g}$$  \hspace{1cm} (33)

which along with (32) gives

$$f(x, y) = B\mathcal{F}_1^{-1}\{|\xi|\mathcal{F}_1 g\}$$  \hspace{1cm} (34)
This form of the Radon inversion is referred to as the *filter back-projection* method and is shown in Figure 6c.
3.3. The Inverse Radon Transform

It can be shown that [5] \( f(x, y) \) can be retrieved from \( g(s, \theta) \) by the inverse Radon transform as follows:

\[
f(x, y) = \frac{1}{2\pi^2} \int_{0}^{\infty} \int_{0}^{\pi} \left[ \frac{\partial g}{\partial s} / [x \cos \theta + y \sin \theta - s] \right] ds \, d\theta
\]

(5)

Proof:

\[
f(x, y) = \int_{0}^{\infty} \int_{0}^{\pi} F(z_{1}, z_{2}) \exp \left[ j 2\pi \left( \frac{x}{s} \cos \theta + \frac{y}{s} \sin \theta \right) \right] \, ds \, d\theta
\]

\[
f(x, y) = \int_{0}^{\infty} \int_{0}^{\pi} F(s \cos \theta, s \sin \theta) \exp \left[ j 2\pi \left( x \cos \theta + y \sin \theta \right) \right] \, ds \, d\theta
\]

\[
f(x, y) = \int_{0}^{\infty} \int_{0}^{\pi} F(s \cos \theta, s \sin \theta) \exp \left[ j 2\pi \left( x \cos \theta + y \sin \theta \right) \right] \, ds \, d\theta
\]

\[
= \int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{\infty} \int_{0}^{\pi} F(s \cos \theta, s \sin \theta) \exp \left[ j 2\pi \left( x \cos \theta + y \sin \theta \right) \right] \, ds \, d\theta
\]

\[
+ \int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi} F(s \cos \theta, s \sin \theta) \exp \left[ j 2\pi \left( x \cos \theta + y \sin \theta \right) \right] \, ds \, d\theta
\]

\[
= \int_{0}^{\infty} \int_{0}^{\pi} F(s \cos \theta, s \sin \theta) \exp \left[ j 2\pi \left( x \cos \theta + y \sin \theta \right) \right] \, ds \, d\theta
\]
\[ I = 0 = \int \int F(-\kappa \theta, -\nu \theta) \exp[-jn \phi(\kappa \cos \theta + \nu \sin \theta)] \, d\phi \, d\theta \]

\[ J = -J \]

\[ I = 0 = \int \int F(s \theta, \nu \theta) \exp\left[ jn \phi(s \cos \theta + \nu \sin \theta) \right] \, ds \, d\theta \]

Applying \( I = 0 \) and \( J = -J \) yields

\[ f(x, y) = \int \int |s| G(s, \theta) \exp\left[ jn \phi(s \cos \theta + \nu \sin \theta) \right] \, ds \, d\theta \]

\[ = \int \int \hat{g}(s \cos \theta + \nu \sin \theta, \theta) \, ds \, d\theta \]

When

\[ \hat{g}(s, \theta) = \int |s| G(s, \theta) \exp\left[ j2n \phi s \right] \, ds \]
Writing

\[ |f| G(f, \theta) = \frac{1}{s} G(f, \theta) \sin(f) \]

and applying the convolution theorem, we obtain

\[ \hat{g}(s, \theta) = \int_{-\infty}^{\infty} \left( \frac{1}{s} G(f, \theta) \right) (\sin(f)) \exp(j2\pi f s) \, df \]

\[ = \left[ \mathcal{F}^{-1} \{ s G(f, \theta) \} \right] \otimes \left[ \mathcal{F}^{-1} \{ \sin(f) \} \right] \]

\[ = \left[ \frac{1}{j2\pi n} \frac{\omega g(s, \theta)}{\cos} \right] \otimes \left( \frac{-1}{jns} \right) \]

\[ = \frac{1}{2\pi n^2} \int_{-\infty}^{\infty} \frac{\omega g(t, \theta)}{\cos} \frac{1}{s-t} \, dt \]

which upon substitution in \((?)\) yields

\[ f(u, y) = \frac{1}{2\pi n^2} \int_{-\infty}^{\infty} \frac{\omega g(t, \theta)}{\cos} \frac{1}{s-t} \, dt \, d\theta \]

\[ \alpha e^{\alpha + y e^\theta} \]
Figure 6a.  Inverse Radon transformation using 1-D filtering.

Figure 6b.  Convolution back-projection method for Radon inverse transformation.

Figure 6c.  Filter back-projection method for Radon inverse transformation.
4. FINAL REMARKS

This report has prepared an overview of the fundamentals of the CT and has described various reconstruction techniques reported in the literature. We described three different views for object reconstruction, namely, the back-projection, the Fourier transform, and the inverse Radon transform. The back-projection approach showed that the object reconstruction is equivalent to an image restoration problem with a known PSF of (14). The Fourier-transform method was based on the Projection Theorem which indicted the fact that the 2-D Fourier-transform of the object can be reconstructed from the 1-D Fourier transform of its projections. We also presented the direct method of the inverse Radon transformation for the object reconstruction. The proof of this theorem suggested three reconstruction methods, perhaps the most practical (and widely used) of which is the convolution back-projection method. Finally, it should be emphasized that in all of the developments of this report we have made the assumption that the projections are recorded in the absence of noise. In addition, it has been assumed that the object is still while the projections are being recorded. In a practical situation, however, both noise and motion are present and their effect has to be taken into consideration.

References


