In digital data transmission, we are concerned with the transmission of discrete symbols.

Regardless of whether a source is purely digital or analog that has been converted to digital, it may be advantageous to add or remove redundant digits to the digital signal. This procedure is referred to as coding. (encoder - decoder shown above).
If the digital signals at the modulator input take on one of only two possible values, the communication system is referred to as **binary**.

If one of \( M > 2 \) possible values is available it is referred to as **\( M \)-ary**.

A digital system is referred to as **coherent** if a local reference is available for demodulation which is in phase with the transmitted carrier.

The channel is usually idealized as simply adding white Gaussian noise with double-sided power spectrum density \( \frac{N_0}{2} \) to the signal \( (\text{AWGN}) \)
Consider

\[ n(t): \text{PSD} = \frac{N_0}{2} \]

\[ S(t) \rightarrow \sum \rightarrow Y(t) \rightarrow \text{Receiver} \]

\[ s(t) \]

\[ A \]

\[ -A \]

\[ y(t) \]

Each 1 second pulse is called a bit.
Assume that the starting and ending times of each pulse are known by the receiver.

The task of the receiver is to decide whether the transmitted signal was A or -A during each bit period. One easy method is to pass it through a LPF, sample its output at some time within each T-second interval, and determine the sign of the sample.

A better procedure is to compare the area of the received signal plus noise waveform (data) with zero at the end of each signaling interval by integrating the received data over the T-second signaling interval (as shown).

For obvious reasons, this receiver is referred to as an integrate-and-dump detector.
The question is, how well does this receiver perform, and on what parameters does its performance depend?

A useful criterion of performance is probability of error, \( P_E \), and it is this we now compute.
the output of the integrator at the end of each signaling interval is

\[ V = \int_{t_0}^{t_0 + T} y(t) \, dt = \int_{t_0}^{t_0 + T} \left[ s(t) + n(t) \right] \, dt \]

\[ = \begin{cases} 
AT + N & \text{if } +A \text{ is sent} \\
-AT + N & \text{if } -A \text{ is sent}
\end{cases} \]

where \( N \) is a r.v. defined as

\[ N = \int_{t_0}^{t_0 + T} n(t) \, dt. \]

Since \( N \) results from a linear operation on a sample function from a Gaussian process it is a Gaussian r.v.

\[ E\{N\} = E\left\{ \int_{t_0}^{t_0 + T} n(t) \, dt \right\} = \int_{t_0}^{t_0 + T} E\{n(t)\} \, dt = 0. \]

\[ \text{Var}\{N\} = E\{N^2\} = E\left\{ \left[ \int_{t_0}^{t_0 + T} n(t) \, dt \right]^2 \right\} = \int_{t_0}^{t_0 + T} \int_{t_0}^{t_0 + T} E\{n(t)\} E\{n(t')\} \, dt \, dt' = \frac{N_0 T}{2}. \]
thus
\[ f_N(\eta) = \frac{1}{\sqrt{2\pi N_0 T}} e^{-\eta^2 / N_0 T}. \]

There are two ways in which error occurs.

1. A pent \( T = AT + N < 0 \) or \( N < -AT \)
2. A pent \( T = AT + N > 0 \) or \( N > AT \)

Let's consider 1. From \( \star \) we get:

\[ P(\text{error}|A \text{ limit}) = P(E|A) = P(N < -AT) \]

\[ = \int_{-\infty}^{-AT} \frac{1}{\sqrt{2\pi N_0 T}} e^{-\eta^2 / N_0 T} d\eta = \frac{1}{2} - \int_{-AT}^{0} \frac{1}{\sqrt{2\pi N_0 T}} e^{-\eta^2 / N_0 T} d\eta \]
\[ u = \frac{-\eta}{\sqrt{N_0 T}} \]

\[ P(E|A) = \frac{1}{2} - \int_{-\infty}^{A \sqrt{\frac{T}{N_0}}} e^{-u^2} \left( -\sqrt{\frac{N_0 T}{2\pi}} \right) du. \]

\[ = \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{A \sqrt{\frac{N_0}{N_0}}} e^{-u^2} du. \]

**Recall:** \[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-y^2} dy \quad \text{;} \quad \text{erf}(x) = 1 - \text{erf}(-x) \]

\[ P(E|A) = \frac{1}{2} - \frac{1}{2} \text{erf} \left( A \sqrt{\frac{T}{N_0}} \right). \]

\[ = \frac{1}{2} \text{erfc} \left( A \sqrt{\frac{T}{N_0}} \right). \]

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt \]

\[ \text{erf}(x) = Q(\sqrt{2}x) \]

\[ P(E|A) = Q \left( \frac{A \sqrt{T}}{\sqrt{N_0}} \right) = Q \left( \frac{\sqrt{2AT}}{\sqrt{N_0}} \right) \]
new let's consider the. From it follows that

\[ P(E1-A) = \int_{\text{AT}}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{N_0T}} e^{-\frac{\eta^2}{2N_0T}} d\eta. \]

\[ \text{AT} \]

\[ \text{-AT} \quad \text{AT} \]

This is equal to what we found for \( P(E1A) \).

\[ P(E1-A) = \frac{1}{2} \text{erfc} \left( A\frac{T}{N_0} \right). \]

The average path of error is

\[ P_E = P(E, A) + P(E, -A) \]

\[ P_E = P(E1A) P(A) + P(E1-A) P(-A) \]

\[ = P(E1A) \left( \frac{1}{2} \text{erfc} \left( A\frac{T}{N_0} \right) \right) \]

\[ = P(E1A) \left( \frac{1}{2} + \text{erfc} \left( A\frac{T}{N_0} \right) \right). \]
We notice that the important factor is $A\sqrt{\frac{T}{N_0}}$.

Let $Z = \frac{A^2 T}{N_0}$. We can interpret this ratio in two ways.

1. $E_s = \int_{t_0}^{t_0+T} A^2 \, dt = A^2 T \quad \leftarrow$ energy in each signal

2. $Z = \frac{A^2 T}{N_0} = \frac{E_s}{N_0} = \frac{\text{signal energy per pulse}}{\text{noise power spectral density}}$

2. Recall that

\[
\begin{align*}
B_p &= \frac{1}{T} \quad \text{is a rough measure of the B.W.} \\
Z &= \frac{A^2}{N_0 \left( \frac{1}{T} \right)} = \frac{A^2}{N_0 B_p} = \frac{\text{signal power per pulse}}{\text{noise power in the signal bandwidth}}
\end{align*}
\]
\[ z \text{ is referred to as a Signal-To-Noise Ratio (SNR).} \]

\[ \begin{align*}
\text{A very good approximation for } \text{erfc}(u) & \text{ is:} \\
\text{erfc}(u) & = \frac{e^{-u^2}}{u\sqrt{\pi}}, \quad u \gg 1 \\
\end{align*} \]

\[ \begin{align*}
P(E) & = \frac{e^{-z}}{z\sqrt{\pi}z}, \quad z \gg 1 \\
\end{align*} \]
Digital data is to be transmitted through a baseband system with $N_0 = 10^{-7}$ W/Hz and the received signal amplitude $A = 20$ mv.

(a) If $10^3$ bit per second (bps) are transmitted, what is $p_x$?

$$T = 1 \text{ ms} = \frac{p}{e} \quad P_e \approx \frac{e^{-\frac{z}{2\sqrt{h\eta}}}}{2\sqrt{h\eta}} \quad z \gg 1$$

$$z = \frac{A^2 T}{N_0} = \frac{(0.02)^2 (10^3)}{10^{-7}} = 4$$

$$\therefore \quad P_e \approx 2.58 \times 10^{-3}$$

(b) If $10^6$ bps are transmitted, to what value must $A$ be adjusted in order to attain the same $p_x$ as in (a).

$$z = \frac{A^2 (10^{-4})}{10^{-7}} = 4 \quad \Rightarrow \quad A = 63.2 \text{ mv}$$
Detection of Binary Signals in AWGN (General):

In the previous section, we analyzed a simple
baseband digital communication system. As in
the case of analog transmission, it is often
necessary to utilize modulation to condition a
digital message signal so that it is suitable
for transmission through a channel. Thus, we
shall let a logic 1 be represented by \( s_1(t) \) and
a logic 0 by \( s_2(t) \). The restriction on \( s_1(t) \) and
\( s_2(t) \) is that they must have finite energy
in a \( T \)-second interval.

\[
E_1 = \left. \int_{t_0}^{t_0+T} s_1^2(t) \, dt \right| < \infty
\]

\[
E_2 = \left. \int_{t_0}^{t_0+T} s_2^2(t) \, dt \right| < \infty
\]
possible choices for $s_1(t)$ and $s_2(t)$ are as follows:

<table>
<thead>
<tr>
<th>Case</th>
<th>$s_1(t)$</th>
<th>$s_2(t)$</th>
<th>Type of Signaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0</td>
<td>$A \cos wt$</td>
<td>ASK (Amplitude-Shift Keying).</td>
</tr>
<tr>
<td>2.</td>
<td>$A \sin(\omega t + \cos m)$</td>
<td>$A \sin(\omega t - \cos m)$</td>
<td>PSK (Phase-Shift Keying with carrier (\cos m = modulation index)</td>
</tr>
<tr>
<td>3.</td>
<td>$A \cos wt$</td>
<td>$A \cos(\omega_0 + \Delta \omega)t$</td>
<td>FSK (Freq-shift Keying).</td>
</tr>
</tbody>
</table>

A possible receiver structure for detecting $s_1(t)$ or $s_2(t)$ in AWGN is shown below:

![Receiver Structure Diagram]

AWGN: $n(t)$
Set IC = 0 each T seconds

Decision
\[ y(t) = s_1(t) + n(t) \]

or
\[ y(t) = s_2(t) + n(t) \]

let \( t_0 = 0 \) and assume that \( s_1(t) \) and \( s_2(t) \) were chosen such that \( s_1(T) = s_1 < s_2(T) = s_2 \).

Now, the decision strategy is as follows:

If \( V > A \), decide that \( s_1(t) \) sent.

If \( V < A \), decide that \( s_2(t) \) sent.

\[ V(t) = \int_{-\infty}^{\infty} y(\lambda) h(t-\lambda) d\lambda = 0 \]

\[ V = \int_{-\infty}^{\infty} y(\lambda) h(T-\lambda) d\lambda \]

Notice that an error occurs in either one of two ways.

1. \( s_1(t) \) sent and \( V > A \)
2. \( s_2(t) \) sent and \( V < A \)
Also note that

\[ V = \begin{cases} 
S_1 + N & \text{if } S_1(t) \text{ present} \\
S_2 + N & \text{if } S_2(t) \text{ present}
\end{cases} \]

where \( N = \frac{n_0(t)}{t = T} \)

We now find the characteristics of \( N \).

Because the filter is fixed linear, \( n_0(t) \) is Gaussian and stationary.

\[ E\{ N \} = 0 \]

\[ S_{n_0}(f) = \left| H(f) \right|^2 \frac{n_0}{2} \]

\[ \sigma^2 = \int_{-\infty}^{\infty} \left| H(f) \right|^2 \frac{n_0}{2} df = n_0 \int_{0}^{\infty} \left| H(f) \right|^2 df \]

\[ f_N(\eta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\eta^2/2\sigma^2} \]
Given that $S_1(t)$ is present, the PDF of $V$ is

$$f_V(v | S_1(t)) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(v-S_1)^2}{2\sigma^2}}$$

and given that $S_2(t)$ is present, the PDF of $V$ is

$$f_V(v | S_2(t)) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(v-S_2)^2}{2\sigma^2}}$$

$P(E|S_2) = \Pr(S_2 < A | S_2(t))$

$P(E|S_1) = \Pr(S_1 > A | S_1(t))$
\[ P(E|S_i) = \int_{-\infty}^{\infty} f_v (v|S_i) \, dv \]

\[ P(E|S_s) = \int_{-\infty}^{A} f_v (v|S_s) \, dv \]

Let
\[ P[S_i (t) \text{ true}] = p \]
\[ P[S_s (t) \text{ true}] = q \]

\[ P_E = P P[S_i | S] + q P[S_s | S] \]
\[ = P \int_{-\infty}^{A} f_v (v|S_i) \, dv + q \int_{-\infty}^{A} f_v (v|S_s) \, dv \]

\[ \frac{dP_E}{dA} = 0 \implies A_{\text{opt}} \]

\[ P_E = P \int_{A}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{v-S_i}{\sigma}\right)^2} \, dv + q \int_{-\infty}^{A} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{v-S_s}{\sigma}\right)^2} \, dv \]
\[
\frac{dP_E}{dA} = -P \frac{1}{\sqrt{2\pi} \sigma^2} e^{-\frac{(A-S_1)^2}{2\sigma^2}} + \frac{4}{\sqrt{2\pi} \sigma^2} e^{-\frac{(A-S_2)^2}{2\sigma^2}} = 0
\]

\[
P = \frac{e^{-\frac{(A-S_1)^2}{2\sigma^2}}}{e^{-\frac{(A-S_1)^2}{2\sigma^2}}} = e^{-\left(\frac{(A-S_1)^2}{2\sigma^2} - \frac{(A-S_2)^2}{2\sigma^2}\right)}
\]

\[
\ln \frac{P}{q} = -\left[\frac{(A-S_2)^2 - (A-S_1)^2}{2\sigma^2}\right]
\]

\[
= \frac{1}{2\sigma^2} \left[-S_2^2 + 2AS_2 + S_1^2 - 2AS_1\right]
\]

\[
= \frac{1}{2\sigma^2} \left[(S_1^2 - S_2^2)(S_1 + S_2) - 2A(S_1 - S_2)\right]
\]

\[
= \frac{S_1 - S_2}{2\sigma^2} \left[S_1 + S_2\right] - \frac{2A(S_1 - S_2)}{2\sigma^2} A_{opt}
\]

\[
A_{opt} = \left(-\ln \frac{P}{q}\right) \frac{\sigma^2}{S_1 - S_2} + \frac{S_1 + S_2}{2}.
\]

Or
\[
A_{opt} = \frac{\sigma^2}{S_1 - S_2} \ln \frac{P}{q} + \frac{S_1 + S_2}{2}.
\]
If \( P = \frac{1}{2} \),

then \( A_{opt} = \frac{S_1 + S_2}{2} \).

With this choice of \( A \), \( P_E \) reduces to

\[
P_E = \frac{1}{2} \text{erfc} \left[ \frac{S_2 - S_1}{2\sqrt{2}\sigma} \right].
\]

Also,

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{v^2}{2}} \, dv.
\]

It can be shown that

\[
\frac{1}{2} \text{erfc}(x) = Q(\sqrt{2}x)
\]

\[
P_E = Q \left( \frac{S_2 - S_1}{2\sigma} \right).
\]

Note that \( P_E \) depends on the dissimilarity of the output signals, \( S_1 \) and \( S_2 \), at \( t = T \).
Notice that $P_E$ is minimized with the choice of

$$A = \frac{\delta^2}{\sigma_{\text{opt}}} \ln \frac{P}{q} + \frac{\sigma_1^2 + \sigma_2^2}{2}.$$

Now, we wish to find the optimum value of $h(t)$ that results in minimum $P_E$. This will lead us to the matched filter.
So far we have shown that

\[ P_E = Q\left( \frac{S_2 - S_1}{20}\right) \]

We now wish to minimize \( P_E \) with respect to the filter \( H(f) \). This can be done by maximizing \( \sum (f) \) w.r.t. \( H(f) \) to find \( H_0(f) \). We note that

\[ \hat{S}(f) = \sum (f) H(f) \]

and

\[ S_2 = \int_{-\infty}^{\infty} H(f) \hat{S}(f) e^{j2\pi f t} df \]

\[ S_1 = \int_{-\infty}^{\infty} H(f) \hat{S}(f) e^{j2\pi f T} df \]

Therefore

\[ \hat{S}_i(f) = \sum_1^T \hat{S}_i(f) \quad i=1,2 \]

Also

\[ \hat{S}(f) = \begin{cases} 0 & \text{if } f \notin \sum_1^T \hat{S}_i(f) \end{cases} \]

\[ \sigma^2 = \frac{N_o}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \]
\[ s^2 = \left( \frac{\hat{S}_2 - \hat{S}_1}{\sigma_0} \right)^2 \]

\[ f^2 = \frac{\int_{-\infty}^{\infty} \left( \hat{S}_2(f) - \hat{S}_1(f) \right) e^{j2\pi f T} df \right)^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \]

It can be shown, using Schwartz's inequality, that

\[ H_0(f) = \left[ \hat{S}_x^*(f) - \hat{S}_y^*(f) \right] e^{-j2\pi f T} \]

For which

\[ f_{\text{max}}^2 = \frac{2}{N_0} \int_{-\infty}^{\infty} \left| \hat{S}_x(f) - \hat{S}_y(f) \right|^2 df = \frac{2}{N_0} \int_0^T (\hat{S}_2(t) - \hat{S}_1(t))^2 dt \]

\[ = \frac{2}{N_0} \left[ E_{\hat{S}_2} + E_{\hat{S}_1} - 2 E_{\hat{S}_2} \hat{S}_1 \right] \quad ; \quad E_{\hat{S}_j} = \int_0^T \hat{S}_j(t) \hat{S}_j(t) dt \]

Also,

\[ h_0(t) = \hat{S}_2(T-t) - \hat{S}_1(T-t) \]
\[ P_{E_{\text{min}}} = Q\left( \frac{f_{\text{max}}}{2} \right) \]

\[ = Q\left( \frac{\sqrt{\frac{E}{N_0} \left( E_{s2} + E_{s1} - 2E_{e1} \right)}}{\frac{4}{2}} \right) \]

\[ = Q\left( \sqrt{\frac{E_{s2} + E_{s1} - 2E_{e1}}{2N_0}} \right) \]

---

Note that \( E_{e1} \) may be a negative number! (e.g., \(+A\))

Note that \( P_E \) can be affected

In further minimization, i.e.,

Let \( E_{e1} = 0 \); i.e.,

by letting \( s_1(t) \) and \( s_2(t) \) be

\( s_1(t) \) and \( s_2(t) \) be

This is referred to

as a target signal.
Remarks

1. \[ y(t) \xrightarrow{s(T-t)} s(T) \xrightarrow{\times} u(T) \]

\[ u(t) = \int_{-\infty}^{\infty} y(\tau) s(T-(t-\tau)) \, d\tau \]

\[ u(t) = \int_{-\infty}^{T} y(\tau) s(T-\tau+T) \, d\tau \]

\[ u(t) = \int_{-\infty}^{T} y(t) s(t) \, dt \]

A matched filter is equivalent to correlation receiver at \( t = T \).
Let the input of a matched-filter be as shown.

![Signal Waveform](image)

a: Find the impulse response of the matched-filter.

b: Obtain the output of this filter.

c: Graph the output of the correlator processor.
Ex. Let the input of a matched filter be as shown. 

(a) Find the matched filter impulse response. (Let $T = 3$).
(b) Obtain the output of this filter.

$$h(t) = s(T-t) = s(-t-T) = s(-t)|_{t \to t-T}$$

Note that if $T < 3$, the filter is non-rectangular. The smaller $T$ is, the nearer we are to want to be the peak occurs.

Max at $t = T = \frac{1}{2}$ and also $\max = E_s^\frac{1}{2} = \frac{1}{2}$.
Note that the output of the correlator is

\[ \int_{0}^{t} s^2(t) \, dt \]

and it is as shown:

Clearly, at \( t = T = 3 \) the output of the correlator and the matched filter are the same.
2. We showed that the optimum filter has the form:

\[ h_0(t) = \frac{1}{2} \delta(T-t) - \delta(T-t) \]
1. Antipodal Baseband Signaling

\[ S_1(t) = -A \]
\[ S_2(t) = A \]

\[ P_E = Q \left( \frac{\sqrt{E_b} + \frac{E_s}{2} - 2E_b}{2N_0} \right) = Q \left( \sqrt{\frac{E_b}{2N_0}} \right) \]

\[ E_{x_2} = E_{s_1} = A^2 T = E_b \]

\[ E_{x_1} = \int_0^T (-A)^2 dt = -\int_0^T A^2 dt = -E_b \]

\[ P_E = Q \left( \sqrt{\frac{E_b + E_b + 2E_b}{2N_0}} \right) = Q \left( \sqrt{\frac{2E_b}{N_0}} \right) \]
2. Biphase shift-keyed signaling (BPSK)

\[ s_1(t) = -A_c \cos 2\pi f_o t \]
\[ s_c(t) = A_c \cos 2\pi f_o t \]

\[ P_E = Q \left( \sqrt{\frac{E_{s_1} + E_{s_c} - 2E_{s_1}}{2N_0}} \right) \]

\[ E_{s_1} = E_{s_c} = \int_0^T (A_c \cos 2\pi f_o t)^2 \, dt \]

\[ = \frac{A_c^2}{2} \int_0^T (1 + \cos 4\pi f_o t) \, dt = \frac{A_c^2 T}{2} = E_b \]

\[ E_{s_1} = -E_{s_1} = -E_b \]

\[ : p_E = Q \left( \sqrt{\frac{4E_b}{2N_0}} \right) = Q \left( \sqrt{\frac{2E_b}{N_0}} \right) \]
3. Amplitude shift - keying (ASK) signaling

\[ s_1(t) = 0 \]

\[ s(t) = A_c \cos 2\pi f_c t \quad \text{for} \quad 0 \leq t \leq T_b \]

\[ P_e = Q\left(\sqrt{\frac{E_s + E_s - 2E_v}{2N_0}}\right) \]

\[ E_b = \frac{E_s + E_s}{2} = \frac{1}{2} E_s \]

\[ E_s = zE_b \]

\[ E_{r_1} = 0 \]

\[ P_e = Q\left(\sqrt{\frac{zE_b}{2N_0}}\right) = \frac{1}{2} \]

\[ \text{ASK requires twice as much energy as BPSK to produce the same } P_e. \]

\[ \text{or: BPSK is } 3 \text{ dB better than ASK in terms of } \text{SNR required for a given } P_e. \]

\[ (\text{see CH} \cdot 3 - 3) \]
4. **Coherent freq shift-keyed (FSK) signal**

\[ s_1(t) = A_c \cos 2\pi f_o t \]
\[ s_2(t) = A_c \cos 2\pi (f_o + \Delta f) t \]
\[ \Delta f = \frac{m}{2T_b} \]

\[ P_E = Q\left( \sqrt{\frac{E_{s_2} + E_{s_1} - 2E_{s_1}}{2N_0}} \right) \]

\[ E_{s_1} = 0 \quad s_1(t) \text{ and } s_2(t) \text{ are coherently orthogonal.} \]

\[ E_{s_2} = E_{s_1} = E_b \]

\[ \therefore P_E = Q\left( \sqrt{\frac{2E_b}{2N_0}} \right) = Q\left( \sqrt{\frac{E_b}{N_0}} \right) \]

Coherent FSK signaling has the same performance as ASK in terms of \( \frac{E_b}{N_0} \) to provide a just

\[ P_E \quad \left( \approx \frac{3-5}{16} \right) \]
EXAMPLE 3-2

Find $E_b/N_0$ in decibels to give a $P_e$ of (a) $10^{-4}$, (b) $10^{-5}$, (c) $10^{-6}$ for (i) baseband and BPSK, (ii) ASK and FSK.

**Solution:** Using (3-35), we have for baseband and BPSK that

$$P_{E,BPSK} = \frac{\exp(-E_b/N_0)}{2\sqrt{\pi E_b/N_0}}$$

An iterative solution using a calculator results in the values in the table that follows for $E_b/N_0$ to give the stated probabilities of error. ASK and FSK require 3.01 dB more in terms of $E_b/N_0$ to give corresponding $P_e$'s.

<table>
<thead>
<tr>
<th>$P_e$</th>
<th>Baseband and BPSK</th>
<th>ASK and FSK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>8.43</td>
<td>11.44</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>9.61</td>
<td>12.62</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>10.54</td>
<td>13.55</td>
</tr>
</tbody>
</table>

EXAMPLE 3-4

Find null-to-null bandwidth of $R_b$ bits per second assuming BPSK. Assume that the received carrier power spectrum is $P_c = 1$ kilobit per second.

**Solution:** For $P_e = 10^{-6}$, so that

$$P_e = \frac{1}{2}\int_0^{\infty} f \left( \frac{1}{2} \right) e^{-2\pi^2 f^2} df$$

where $R_b = \frac{1}{T}$

For $R_b = 10$ kHz,

$$f = \frac{1}{T} = \frac{1}{10^{-4}} = 10^4 MHz$$

at baseband.

$$A_c \cos(2\pi f t)$$

$$A_c \Pi \left( \frac{t}{T} \right)$$
Example 3-4: Find null-to-null BW for baseband BPSK, ASK, and FSK for a data rate of \( R_b = \frac{1}{T_b} \) bps.

\[ BW = \frac{1}{T_b} = R_b \]

\[ y(t) = A_c H \left( \frac{t}{T_b} \right) \cos(2\pi f_c t) \]

\[ F\{ y(t) \} = \frac{A_c T_b}{2} \left\{ \sin\left[ T_b (f - f_c) \right] + \sin\left[ T_b (f + f_c) \right] \right\} \]

\[ BW = \frac{2}{T_b} = 2R_b \]
c. FSK : For FSK we have two carriers:
\[ A_c \cos 2\pi f_c t \text{ and } A_c \cos \left[ 2\pi (f_c + \Delta f) t \right] \].

\[ \text{BW} = 2R_b + \Delta f \]

\[ f_c \quad f_c + \Delta f \]

\[ \Delta f \quad 2R_b \]

\[ f \]

d. ASK : 
\[ s_i(t) = 0, \quad s_e(t) = A_c \cos 2\pi f_c t \]

\[ y(t) = \Pi \left( \frac{t}{T_b} \right) A_c \cos 2\pi f_c t \text{ or } 0 \]

This is similar to BPSK.

\[ \text{BW} = 2R_b \]

<table>
<thead>
<tr>
<th>Modulation</th>
<th>Bandwidth</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>BW = 2R_b</td>
</tr>
<tr>
<td>FSK</td>
<td>BW = 2R_b + \Delta f</td>
</tr>
<tr>
<td>ASK</td>
<td>BW = 2R_b</td>
</tr>
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<td>BPSK</td>
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</tr>
<tr>
<td>ASK</td>
<td>BW = 2R_b</td>
</tr>
</tbody>
</table>
Example 3.5: \[ N_0 = 10^{-8} \frac{W}{Hz} \]

Channel \[ B = 1 \text{ MHz} \]

Consider ASK, BPSK, and FSK.

(a) What is the maximum data rate that can be supported by the channel?

**Sol:** ASK and BPSK \[ B = 2R_b \rightarrow \frac{R_b}{B} = \frac{1}{2} \]

\[ R_{b_{max}} = \frac{1 \text{ Mbps}}{2} = 500 \text{ Kbps} \]

For FSK, \[ B = 2.5R_b \rightarrow R_b = 400 \text{ Kbps} \]

(b) Find the received signal power required to give \[ P_E = 10^{-6} \] at the data rate found in (a).

**Sol:** Known for BPSK \[ P_E = Q\left(\frac{\sqrt{2E_b}}{N_0}\right) = 10^{-6} \]

From page 162 \[ \frac{E_b}{N_0} = 10.5 \text{ Y dB = 11.32} \]

For ASK and FSK \[ P_E = Q\left(\frac{\sqrt{E_b}}{N_0}\right) = 10.5 + 3 \]

\[ \text{Check:} \]
Recall from BPSK that
\[
\frac{E_b}{N_0} = \frac{A_c^2 T_b}{2 N_0} = 11.32
\]
\[
\bar{P}_R = \left[ \frac{1}{2} \int_{-T/2}^{T/2} A_c^2 \cos^2(2 \pi f_c t) \, dt \right] = \frac{A_c^2}{4}
\]
\[
\therefore \quad \frac{A_c^2}{2} = (11.3) \left( \frac{N_0}{T_b} \right) = 56.6 \text{ mW}
\]

For ASK:
\[
\bar{P}_R = \left[ \frac{1}{2} \int_{-T/2}^{T/2} A_c^2 \cos^2(2 \pi f_c t) \, dt \right] = \frac{A_c^2}{4}
\]
\[
\frac{E_b}{N_0} = \frac{A_c^2 T_b}{4 N_0} = 0 \quad \frac{A_c^2}{4} = N_0 \left( \frac{E_b}{N_0} \right) \left( \frac{1}{T_b} \right) = 113.3 \text{ mW}
\]
FSK:  \[ \frac{\bar{P}}{R} = \frac{A_c^2}{2} = N_0 (\frac{E_b}{N_0}) B_b \times 10^5 \]

\[ = 90.6 \text{ mW.} \]

<table>
<thead>
<tr>
<th>Modulation</th>
<th>Rate ( R_b )</th>
<th>Power ( P_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BPSK</td>
<td>( \frac{B}{2} ) = 500 kbps</td>
<td>( \frac{A_c^2}{2} ) = 56.6 mW</td>
</tr>
<tr>
<td>ASK</td>
<td>( \frac{B}{2} ) = 500 kbps</td>
<td>( \frac{A_c^2}{4} ) = 113.3 mW</td>
</tr>
<tr>
<td>FSK</td>
<td>( \frac{B}{2.5} ) = 400 kbps</td>
<td>( \frac{A_c^2}{2} ) = 90.6 mW</td>
</tr>
</tbody>
</table>

\( P_E = 10^{-6} \)

\( B = 1 \text{ MHz} \)

\( N_0 = -8 \text{ W/Hz} \)