Signaling through Bandlimited channels:

An assumption considered so far is that there is no bandwidth restriction imposed. Any filtering carried out in the transmitter, channel, or receiver would result in degradation of system performance. In this section, special signal designs and filter function that avoid this degradation will be discussed. (We consider linear case)

System Model:

\[ t_m = mT + t_d \]

\[ \text{PSD: } G_m(f) \]
$$\chi_s(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t-kT), \quad a_k = \{+1, -1\} \quad \text{during kth signaling interval}.$$ 

$$\chi_t(t) = \chi_s(t) \ast \frac{h}{T}(t) = \sum_{k=-\infty}^{\infty} a_k \delta(t-kT) \ast \frac{h}{T}(t) = \sum_{k=-\infty}^{\infty} a_k \frac{h(t-kT)}{T}$$

$$\chi(t) = \chi_t(t) \ast h_c(t) = \chi_s(t) \ast \frac{h}{T}(t) \ast h_c(t)$$

$$y(t) = \chi(t) + n(t)$$

$$u(t) = y(t) \ast \frac{h}{R}(t) = [\chi(t) + n(t)] \ast \frac{h}{R}(t)$$

$$= \chi(t) \ast \frac{h}{R}(t) + n(t) \ast \frac{h}{R}(t).$$

$$= \chi_s(t) \ast \frac{h}{T}(t) \ast h_c(t) \ast \frac{h}{R}(t) + n(t) \ast \frac{h}{R}(t)$$

Let $A P_r(t-t_d)$ where $A$ is chosen such that $P_r(0) = 1$.
\[ U(t) = \chi_s(t) \ast A P_r(t-t_d) + n_0(t) \]

\[ = \sum_{k} a_k \delta(t-kT) \ast A P_r(t-t_d) + n_0(t) \]

\[ = \sum_{k=-\infty}^{\infty} A a_k P_r(t-t_d-kT) + n_0(t). \]

The max of \( P_r(t-t_d) \) is assumed to occur at \( t = t_d \). Thus, the samples should be taken at time \( t_m = mT + t_d \) in order to sample at the peak of each received pulse.

\[ V_m = U(t_m) = \sum_{k=-\infty}^{\infty} A a_k P_r(t - t_d - kT) + n_0(t_m) \]

\[ = \sum_{k=-\infty}^{\infty} A a_k P_r[(m-k)T] + N_m \]
\[ = A m_p(0) + \sum_{k=-\infty}^{\infty} A m_k p(m-k)T + N_m \]  

*desired signal*  

*undesired signal*  

\[ m = -\infty, -2, -1, 0, 1, 2, \ldots \]  

*inter-symbol interference (ISI)*  

Because \( n(t) \) is Gaussian and the receiver filter is a linear system, \( N_m \) are Gaussian. Furthermore, it will be assumed that \( G_n(f) / H_k(f) \) is sufficiently wideband so that any two separate samples, say \( N_k \) and \( N_m, k \neq m \), are uncorrelated and therefore independent.

Now the question is: *is it possible to*  

\[ \underline{\text{eliminate ISI?}} \]  

In the next section we consider this task.
Designing for Zero ISI:

Nyquist’s Pulse-shaping Criterion

Since the channel is fixed, the goal is to choose \( H_T(f) \) and \( H_R(f) \) to minimize the combined effects of ISI and noise on the decision process. The effect of ISI can be completely negated if it is possible to obtain a received pulse shape, \( P_r(t) \), such that

\[
P_r(nT) = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0 
\end{cases} \quad (**) \]

This condition guarantees zero ISI (see **). The following then, referred to as Nyquist’s pulse-shaping criterion, gives a condition on \( F \{ P_r(t) \} \) which results in a pulse shape having zero ISI property (see **)
Theorem: If \( P_r(f) = \mathcal{F}^{-1} P_r(t) \) satisfies the condition

\[
\sum_{k=-\infty}^{\infty} P_r(f + \frac{k}{T}) = T \quad ; \quad |f| \leq \frac{1}{2T}
\]

then

\[
P_r(nT) = \begin{cases} 
1 & n = 0 \\
0 & n \neq 0
\end{cases}
\]

\( \iff \)

\[
P_r(t) = \int_{-\infty}^{\infty} P_r(f) e^{j2\pi ft} \, df
\]

\[
= \sum_{k=-\infty}^{\infty} \int_{\frac{2k+1}{2T}}^{\frac{2k+1}{2T}} P_r(f) e^{j2\pi ft} \, df + \ldots
\]

\[
= \sum_{k=-\infty}^{\infty} \int_{\frac{2k-1}{2T}}^{\frac{2k-1}{2T}} P_r(f) e^{j2\pi ft} \, df
\]

\[
\therefore \quad P_r(nT) = \sum_{k=-\infty}^{\infty} \int_{\frac{2k-1}{2T}}^{\frac{2k+1}{2T}} P_r(f) e^{j2\pi f nT} \, df
\]

\( \text{let} \quad u = f - \frac{k}{T} \)
\[
P_r(nT) = \sum_{k=-\infty}^{\infty} \int_{-\frac{1}{2T}}^{\frac{1}{2T}} P_r(u + k) e^{j2\pi(nT + u) / T} \, du
\]

\[
= \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left[ \sum_{k=-\infty}^{\infty} P_r(u + k) \right] e^{j2\pi nT u / T} \, du
\]

\[
= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \frac{1}{T} \text{sinc}(nT u) \, du
\]

\[
= \text{sinc}(nT)
\]

\[
\sum_{n=0}^{\infty} \frac{1}{n} = \text{QED}
\]

Note: If \( P_r(t) \) is purely real and has odd symmetry about the nominal cutoff frequency, then \( P_r(t) \) is an even function with a peak at \( t=0 \) and periodic zero values spaced by \( 1/2W \).
A class of spectra, \( P_r(f) \), that satisfy the Nyquist pulse-shaping criterion is the Ranced Cosine family,

\[
P_{rc}(f) = \begin{cases} 
  T & \text{if } |f| \leq \frac{1}{2T} - \beta \\
  \frac{1}{2T} \left[ 1 + \cos \left( \frac{\pi |f| - \frac{1}{2T} + \beta}{\beta} \right) \right] & \text{if } \frac{1}{2T} - \beta \leq |f| \leq \frac{1}{2T} + \beta \\
  0 & \text{if } |f| > \frac{1}{2T} + \beta 
\end{cases}
\]
optimum \ H(t) and \ H_R(f) :

Recall that

\[ AP_r(t-t_d) = h_T(t) \times h_c(t) \times h_R(t) \]

\[ A P_r(f) e^{-j 2 \pi f t_d} = H(f) H(f) H(f) \]

\[ A P_r(f) \]

For zero ISI \( P_r(f) \) must satisfy

\[ \sum_{k=\infty}^{\infty} P_r(f + \frac{k}{T}) = T ; \quad \text{if} \ 1 \leq \frac{1}{2T} \]

(2) in (1) yields a constraint between \( H(f) \) and \( H_R(f) \) (note that \( H_c(f) \) is fixed).

A additional constraint on \( H(f) \) and \( H_R(f) \) is that they be chosen such that the prob of making a decision error at the receiver be minimized.
For zero-ISI we have:
\[ V_m = A a_m + N_m \]

\[ P(\varepsilon) = P(V_m > 0 | a_m = -1) \]

\[ = P(-A + N_m > 0) = P(N_m > A) \]

(We have assumed that \( P(a_m = 1) = P(a_m = -1) \)).

But \( n(t) \) is a Gaussian with zero mean.

\[ E \{ N_m \} = 0 \]

\[ \text{Var} \{ N_m \} = \sigma^2 = \int_{-\infty}^{\infty} G_n(f) |H_R(f)|^2 df \]

\[ f_{N_m}(n_m) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{n_m^2}{2\sigma^2}} \]

\[ P(\varepsilon) = \int_{A}^{\infty} \frac{\exp(-u^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}} du = \Phi \left( \frac{A}{\sigma} \right) \]


\[ P(e) = \alpha \left( \frac{\theta}{2} \right) \]

\[ \text{or} \quad \frac{\sigma}{A} \]

\[ \text{or} \quad \frac{\nu}{A^4} \]

Note that the energy of the \( k \text{th} \) transmitted pulse is:

\[ E_T = E \left\{ \int_{-\infty}^{\infty} a_k h_T(t-kt) \, dt \right\} = E \left\{ \int_{-\infty}^{\infty} p_k \int_{-\infty}^{\infty} h_T(t-kt) \, dt \right\} \]

\[ = \int_{-\infty}^{\infty} \left| H_T(f) \right|^2 \, df \]

\[ \text{Parserval's theorem} \]

But

\[ A \cdot P_r(f) e^{-j2\pi ft_0} = H_T(f) H_c(f) H_R(f) \]

\[ A^2 \left| P_r(f) \right|^2 = \left| H_T(f) \right|^2 \left| H_c(f) \right|^2 \left| H_R(f) \right|^2 \]

\[ \therefore \left| H_T(f) \right|^2 = \frac{A^2 \left| P_r(f) \right|^2}{\left| H_c(f) \right|^2 \left| H_R(f) \right|^2} \]

\[ E_T = A^2 \int_{-\infty}^{\infty} \frac{\left| P_r(f) \right|^2}{\left| H_c(f) \right|^2 \left| H_R(f) \right|^2} \, df \quad (T) \]
Recall that
\[
\sigma^2 = \int_{-\infty}^{\infty} G_n(f) |H_K(f)|^2 \, df
\]
and from (+)
\[
A^2 = \frac{E_T}{\int_{-\infty}^{\infty} \frac{|P_r(f)|^2}{|H_c(f)|^2 |H_K(f)|^2} \, df}
\]
\[
= \frac{\sigma^2}{A^2} = \frac{1}{E_T} \int_{-\infty}^{\infty} G_n(f) |H_K(f)|^2 \, df \int_{-\infty}^{\infty} \frac{|P_r(f)|^2}{|H_c(f)|^2 |H_K(f)|^2} \, df.
\]
which is to be minimized through appropriate choice of $H_K(f)$. This can be accomplished through application of Schwarz's inequality:

Recall:
\[
\left| \int_{-\infty}^{\infty} X(f) \overline{Y(f)} \, df \right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^2 \, df \int_{-\infty}^{\infty} |Y(f)|^2 \, df
\]
with equality if \[ |X(f) = \sigma^2 |Y(f)| \]
Let
\[ |X(f)| = C_n(f) |H_R(f)|^{1/2} \]
\[ |Y(f)| = \frac{|P_r(f)|}{|H_c(f)| |H_R(f)|} \]

\[ \int_{-\infty}^{\infty} G_n(f) |H_R(f)|^2 \, df \int_{-\infty}^{\infty} \frac{|P_r(f)|^2}{|H_c(f)| |H_R(f)|^2} \, df \geq \frac{1}{E_T} \left[ \int_{-\infty}^{\infty} \frac{G_n(f) |P_r(f)|}{|H_c(f)|} \, df \right]^2 \]

\[ \frac{\sigma^2}{A^2 \left| \min \right.} \]

with equality iff \[ |X(f)| = \alpha^2 |Y(f)| \]
or

\[ C_n(f) |H_R(f)| = \alpha^2 \frac{|P_r(f)|}{|H_c(f)| |H_R(f)|} \]

\[ \cdot \]

\[ |H_R(f)| = \alpha \frac{|P_r(f)|^{1/4}}{G_n(f) |H_c(f)|^{1/2}} \]

\[ \cdot \]

\[ \text{Also we know:} \quad A \phi_r(f) e^{-j2\pi ft} = H_T(f) H_c(f) H_R(f) \]

\[ \cdot \]

\[ |H_T(f)| = \frac{A |P_r(f)|^{1/4}}{|H_c(f)| |H_R(f)|} \]

\[ \cdot \]

\[ \text{Note that} \quad H_T \text{ and } H_c \text{ are both part of} \]

\[ \text{P}_r(f) \text{ that would satisfy} \phi_r(t) \text{ and} \]

\[ H_c(f) \text{ is used to and} \]

\[ \text{minimax process} \]

\[ \cdot \]

\[ \cdot \]

\[ \cdot \]
Note that

\[
\frac{\sigma^2}{A^2} \bigg|_{\text{min}} = \frac{1}{\mathcal{E}_T} \left[ \int_{-\infty}^{\infty} \frac{G_n^{1/2}(f) |P_r(f)|}{|H_c(f)|} \, df \right]^2
\]

and

\[
P_{\text{min}}(\epsilon) = \mathcal{O} \left( \frac{A}{\epsilon} \right) = \mathcal{O} \left\{ \sqrt{\mathcal{E}_T} \left[ \int_{-\infty}^{\infty} \frac{G_n^{1/2}(f) |P_r(f)|}{|H_c(f)|} \, df \right]^{-1} \right\}.
\]

**Special cases**

1. \( G_n(f) = \frac{N_0}{2} |H_c(f)|^2 \)

\[
|H_r(f)| = \frac{\alpha |P_r(f)|^{1/2}}{G_n^{1/4}(f) |H_c(f)|^{1/2}} = \beta' \frac{|P_r(f)|^{1/2}}{|H_c(f)|}.
\]

and

\[
|H_T(f)| = \frac{(A_0)^{1/4} |P_r(f)|^{1/2} G_n^{1/4}(f)}{|H_c(f)|^{1/2}} = \beta'' |P_r(f)|^{1/2}.
\]

where \( \beta' \) and \( \beta'' \) are arbitrary constants.
\[ P_{\text{min}}(\epsilon) = \mathcal{Q} \left\{ \sqrt{E_T} \left[ \int_{-\infty}^{\infty} \left( \frac{N_0}{2} \right)^{1/2} \frac{1}{H_c(f)} \right]^{-1} \right\} \]

\[ = \mathcal{Q} \left\{ \sqrt{E_T} \left( \frac{N_0}{2} \right)^{1/2} \int_{-\infty}^{\infty} |P_r(f)| \, df \right\}^{-1} \]

Recall that \( P_r(0) = 1 \)

\[ P_r(t) = \int_{-\infty}^{\infty} P_r(f) e^{-j \inf \sqrt{f} \, t} \, df = \int_{-\infty}^{\infty} P_r(f) \, df = 1 \]

Therefore if \( P_r(f) \) is real and positive, we get:

\[ P_{\text{min}}(\epsilon) = \mathcal{Q} \left( \sqrt{\frac{2E_T}{N_0}} \right) \]

which is identical to \( P(\epsilon) \) for matched filter detection of binary signals in AWGN.
\[ |H_c(f)| = H_0 \quad \leftarrow \text{ideal case with no filtering (ideal channel)} \]

\[ Q_n(f) = \frac{N_0}{2} \quad \text{white noise} \]

\[ |H_R(f)| = \gamma' |P_R(f)|^{1/2} \]

\[ |H_T(f)| = \gamma'' |P_R(f)|^{1/2} \]

\[ P_{\min}(\varepsilon) = \mathcal{O} \left\{ \sqrt{E_T} \left[ \int_{-\infty}^{\infty} \frac{\sqrt{N_0}}{H_0} |P_R(f)| \, df \right]^{-1/2} \right\} \]

\[ = \mathcal{O} \left( \sqrt{\frac{2E_T H_0^2}{N_0}} \right) = \mathcal{O} \left( \sqrt{\frac{2E_T}{N_0}} \right) \quad |H_0 = 1| \]

See example 3-6/140
\[
H_c(f) = \frac{1}{1 + j \frac{f}{4880}}
\]

\[
N_0 = 10^{-12} \frac{w}{Hz}
\]

\[
P_r(t) = \beta = \frac{1}{2T} = 4880 \text{ Hz}
\]

Zero ISI and min prob of error.

\[
B = \beta + \frac{1}{2T} = \frac{1}{2T} + \frac{1}{2T} = \frac{1}{T} = 9600 \text{ Hz}
\]

\[
|H_R(f)| = \frac{\alpha \left| P_x(f) \right|^{1/2}}{G_n(f) \left| H_c(f) \right|^{1/2}}
\]

\[
|H_T(f)| = \frac{\left( A/\alpha \right) \left| P_x(f) \right|^{1/2} G_n(f)}{\left| H_c(f) \right|^{1/2}}
\]

\[
G_n(f) = 10^{-12}
\]

From (3-93) and (3-99):

\[
P_{rec}(t) = \frac{C_0 e^{\frac{\beta t}{2}} \cos \left( \frac{t}{T} \right)}{1 - (4\beta t)^2}
\]

\[
|H_R(f)| = \left\{ \begin{array}{ll}
\left[ 1 + \left( \frac{f}{4880} \right)^2 \right]^{1/4} G_0 \left( \frac{4f}{9216} \right), & \text{if } 1 \leq f \leq 9600 \\
0, & \text{elsewhere}
\end{array} \right.
\]

\[
\text{Exa} \frac{3 - \frac{1}{T}}{\text{Text}(210)} \quad \frac{1}{T} = 9600 \quad \text{bps.}
\]
\[ |H_T(f)| = \begin{cases} \sqrt{c_0} \left( \frac{H_T}{19240} \right) & 0 \leq f \leq 9600 \text{ Hz} \\ 0 & \text{elsewhere} \end{cases} \]

\[ P(E) = 0 \left( \frac{A}{0} \right) \leq 10^{-6} \]

\[ (\frac{A}{0}) \geq 4.75 \]

\[ N^{1/2} \]

\[ P_{\text{min}}(E) = Q \left[ \int_{-\infty}^{\infty} \frac{\sqrt{E_T} \left( \frac{G_n(f)}{|H_T(f)|} |P_r(f)| \right)^{-1}}{df} \right] \]

Let \[ F = \int_{-\infty}^{\infty} \frac{|P_r(f)|}{|H_T(f)|} df \approx 1.21 \]

\[ P_{\text{min}}(E) = Q \left[ \frac{1}{1.21} \sqrt{\frac{2E_T}{N_0}} \right] = Q \left\{ 0.83 \sqrt{\frac{2E_T}{N_0}} \right\} \]

For the case of matched filter detection of binary antipodal signals in AWGN we have

\[ P_{\text{min}}(E) = Q \left( \sqrt{\frac{2E_T}{N_0}} \right) \]

The finite BW channel imposes a degradation of

\[ 20 \log_{10} 0.83 = 1.62 \text{ dB over the infinite BW case}. \]
Now \[ \frac{A}{\sigma} = 4.75 \]

and \[ \frac{A}{\sigma} = \frac{1}{1.21} \sqrt{\frac{2E_T}{N_0}} = 1.0 \quad \frac{E_T}{T} = 3.3 \times 10^{-11} \text{ J} \]

\[ N_0 = 2 \times 10^{-12} \]

\[ P = \frac{E_T}{T} = (9682)(3.3 \times 10^{-11}) = 31.68 \text{ MW} \]

\[ 10 \log \left( \frac{31.68}{1 \text{ MW}} \right) = -38 \text{ dBm} \]
Design a binary baseband PAM system to transmit data at a bit rate of 3600 bps with a bit error probability less than $10^{-4}$, where

$$H_c(f) = \begin{cases} 
10^{-2} & |f| < 2400 \\
0 & \text{elsewhere}
\end{cases}$$

and $G_n(f) = 10^{-14}$ W/Hz for all frequencies.
Exa. Design a binary baseband PAM system to transmit data at a bit rate of 3600 \( \frac{1}{s} \) with a bit error prob less that \( 10^{-4} \), when
\[
H_c(f) = \begin{cases} 
10^2 & \text{if } |f| < 2400 \\
0 & \text{elsewhere}
\end{cases}
\]
ideal LPF.

and \( G_n(f) = 10^{-14} \frac{W}{Hz} \) white noise.

As \( \frac{1}{T} = 3600 \), \( Pe \leq 10^{-4} \), and
the channel \( BW = 2400 \frac{Hz}{Hz} \), \( G_n(f) = 10^{-14} \frac{W}{Hz} \).

If we choose a raised cosine pulse spectrum with \( B = \frac{1}{2T} + \beta = 2400 \), then
\( \beta = 2400 - \frac{3600}{2} = 600 \) and the channel constraint is satisfied. Hence,

\[
P_r(f) = \begin{cases} 
\frac{1}{3600} & \text{if } |f| \leq \frac{1}{2T} - \beta = 1200 \\
\frac{1}{3600} \cos^2 \left( \frac{\pi}{2400} (|f| - 1200) \right) & 1200 \leq |f| < \frac{1}{2T} + \beta = 2400 \\
0 & |f| \geq 2400.
\end{cases}
\]

\[ P_r(f) \]
\[ H_c(f) \]
\[ f \]
Recall that for zero ISI we have:

\[ |H_R(f)| = \frac{\alpha \ |P_r(f)|^{1/2}}{G_n(f) \ |H_c(f)|^{1/2}} \]

\[ |H_T(f)| = \frac{(A/\alpha) \ |P_r(f)|^{1/2} G_n(f)}{|H_c(f)|^{1/2}} \]

\[ |H_R(f)| = \frac{\alpha}{(10^{14})^{1/2}} \frac{1}{(10^2)^{1/2}} |P_r(f)|^{1/2} = \gamma' |P_r(f)|^{1/2} \]

\[ |H_T(f)| = \gamma'' |P_r(f)|^{1/2} \]

\[ |H_R(f)| = \begin{cases} \frac{\gamma'}{60} & \text{if } |f| < 1200 \\ \frac{\gamma'}{60} \frac{H}{2400} (|f| - 1200) & \text{if } 1200 \leq |f| < 2400 \\ 0 & \text{if } |f| \geq 2400 \end{cases} \]

To make the gain = 1, let \( \gamma' = 60 \).
notice that \( |H_T(f)| = |H_R(f)| \).

\[
|H_T(f)| = |H_R(f)| = \begin{cases} 
1 & \text{if } |f| < 1200 \\
\frac{100 + f}{2400} & \text{if } 1200 \leq |f| < 2400 \\
0 & \text{if } |f| \geq 2400.
\end{cases}
\]

Now, to maintain a \( P_e \leq 10^{-4} \) we do the following:

\[
P_{\text{min}}(\varepsilon) = Q\left(\frac{A}{\sigma}\right) = Q\left\{\sqrt{\mathbb{E}_T\left[\int_{-\infty}^{\infty} \frac{G_n(f) |P_r(f)|}{|H_c(f)|} \, df\right]^{-1}}\right\}
\]

\[
\leq 10^{-4}
\]

Using the table

\[
\sqrt{\mathbb{E}_T\left[\int_{-\infty}^{\infty} \frac{G_n(f) |P_r(f)|}{|H_c(f)|} \, df\right]^{-1}} \geq 3.75
\]

\[
o_n E_T \geq (3.75)^2 \left[\int_{-\infty}^{\infty} \frac{G_n(f) |P_r(f)|}{|H_c(f)|} \, df\right] \geq \frac{(10^{-14})^{1/2}}{10^{-2}}
\]

\[
= (14.06) (10^{-5}) \left[\int_{-\infty}^{\infty} |P_r(f)| \, df\right]^2 = 14.06 \times 10^{-10}
\]

Joules.
\[
P_T = \frac{E_T}{T} = \frac{14.06 \times 10^{-10}}{1/3600} = 50616 \times 10^{-10} \text{ W.}
\]

\[
\approx -23 \text{ dBm}
\]

which completes the design.

\[\text{dBm} = 10 \log \frac{P_T}{1 \text{ mw}}\]
APPENDIX D
GAUSSIAN PROBABILITIES

(1) \( P(X > \mu_X + y\sigma_X) = Q(y) = \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \)

(2) \( Q(0) = \frac{1}{2} \); \( Q(-y) = 1 - Q(y) \), when \( y \geq 0 \)

(3) \( Q(y) = \frac{1}{y\sqrt{2\pi}} e^{-y^2/2} \) when \( y \gg 1 \) (approximation may be used for \( y > 4 \))

(4) \( \text{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{-z^2} dz = 2Q(\sqrt{2}y), \ y > 0 \).

Figure D.1 Gaussian probabilities.
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$Q(y) = 10^{-3}$

$Q(y) = \frac{10^{-3}}{2}$

$Q(y) = 10^{-4}$

$Q(y) = \frac{10^{-4}}{2}$

$Q(y) = 10^{-5}$

$Q(y) = 10^{-6}$
Correlative Coding (Chap 3 cont.)

This far we have treated intersymbol interference (ISI) as an undesirable phenomenon that produces a degradation in system performance. But, it can be shown that by adding intersymbol interference to the transmitted signal in a controlled manner, it is possible to achieve a signaling rate of \( \frac{2B}{T} \) symbols per second in a channel of \( BW = B_f + H_2 \). Such schemes are called Correlative Coding or Partial-Response Signaling. Schemes

\[
\frac{B}{2} \leq B \leq R
\]

Duobinary Signaling

The basic idea of correlative coding will now be illustrated by considering the specific example of duobinary signaling, where "duo" implies doubling of the transmission capacity of a straight binary system.
Consider a binary input sequence \( \{b_k\} \) consisting of uncorrelated binary digits with symbol 1 represented by a pulse of amplitude +1 V, and symbol 0 by a pulse of amplitude -1 V.

\[
\begin{align*}
\{b_k\} & \rightarrow \oplus \rightarrow H_c(f) \rightarrow \sqrt{t=KT_b} \\
\text{Delay} \frac{T_b}{2} & \rightarrow H(f) \rightarrow \{c_k\}
\end{align*}
\]

\[c_k = b_k + b_{k-1}\]

\(\{c_k\}\) has three levels (-1, 0, +1)

Know that for direct binary \(\{b_k\}\) we have

\[
S_{X_b}(f) = A^2 T \left( \frac{\sin \pi f T}{\pi f T} \right)^2
\]

and

\[
S_{X_c}(f) = A^2 T \left( \frac{\sin 2\pi f T}{2\pi f T} \right)^2
\]
Notes:

- Bandwidth compression due to duality in signaling is seen from the figure.

- $S_{X_c}(f)$

- $S_{X_b}(f)$

- $\frac{2bR}{T}$

- $H(f)$

- $BW = \frac{1}{2T}$

Recall: $B = \frac{1}{2T} + \beta \rightarrow 0 = 0 \quad B_{\text{min}} = \frac{1}{2T}

- Minimum $BW$ can be achieved using duality method.

- If we send the data with the bit rate $\frac{b}{T}$, then $BW = \frac{1}{T}$ which is what is seen for $S_{X_b}(f)$. Then, we can occupy the same $BW$ as $S_{X_b}(f)$ but with twice the bit rate.
\[ H(f) = H_c(f) \left[ 1 + e^{-j2\pi ft_b} \right] \]

\[ = H_c(f) e^{-j\pi ft_b} \left[ e^{j\pi ft_b} + e^{-j\pi ft_b} \right] \]

\[ = 2H_c(f) \cos(\pi ft_b) e^{-j\pi ft_b} \]

For an ideal channel of \( BW = \frac{1}{2T_b} \), we have

\[ H_c(f) = \begin{cases} 
1 & \text{if } |f| \leq \frac{1}{2T_b} \\
0 & \text{otherwise} 
\end{cases} \]

\[ H(f) = \begin{cases} 
2 \cos(\pi ft_b) e^{-j\pi ft_b} & \text{if } |f| \leq \frac{1}{2T_b} \\
0 & \text{otherwise} 
\end{cases} \]

\( h(t) = \mathcal{F}^{-1}\{H(f)\} \)
An advantage of the freq response is that it can be easily approximated in practice.

Notice that the original data \( \{b_k\} \) may be detected from duobinary coded sequence \( \{c_k\} \) as follows:

\[
\hat{b}_k = \hat{c}_k - b_{k-1}
\]

It is apparent that if \( c_k \) is received without error, and if also \( \hat{b}_{k-1} \) corresponds to a correct decision, then \( \hat{b}_k \) will be correct too. The technique of using a stored estimate of the previous symbol is called decision feedback.
A major drawback of the detection process is
that once errors are made, they tend to propagate.

A practical means of avoiding this is to use
precoding before the duo-binary coding:

\[
\begin{align*}
\{b_k\} \rightarrow \{a_k\} \rightarrow \{c_k\}
\end{align*}
\]

where \( c_k = a_k \oplus a_{k-1} \)

\[
\begin{align*}
a_k &= b_k \oplus a_{k-1} \\
\quad \quad \quad = \begin{cases} 
  a_{k-1} & \text{if } b_k = 0 \\
  a_{k-1} & \text{if } b_k = 1 
\end{cases}
\end{align*}
\]

\( C_k \) depends on \((b_k)\) only and does not
propagate the error.

\( C_k \) addition is equivalent to Exclusive-OR
operation: the output of a two-input Exclusive-OR
gate is a 1 if exactly one input is a 1; otherwise,
the output remains a 0.
Assume that for $\{a_k\}$ we have:

<table>
<thead>
<tr>
<th>symbol</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
C_k &= \begin{cases} 
q_{k+1} + a_{k-1} = \pm 2 \text{ v}, & b_k = 0 \\
q_{k+1} - a_{k-1} = 0 \text{ v}, & b_k = 1 \\
\text{or } & -1 + 1 \\
1 + 1 &
\end{cases}
\end{align*}
\]

From above we deduce the following decision rule for detecting $\{b_k\}$ from $\{C_k\}$:

\[
b_k = \begin{cases} 
symbol 0, & \text{if } |C_k| > 1 \text{ volt} \\
symbol 1, & \text{if } |C_k| < 1 \text{ volt}
\end{cases}
\]

\[
\{C_k\} \xrightarrow{\text{Rectifier}} \{|C_k|^2\} \xrightarrow{\text{Decision}} \{b_k\}
\]

\text{decision depends only on } C_k
\[
\begin{align*}
\{b_k\} & \quad \begin{array}{cccccccc}
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
\{a_k\} & = & b_k & \theta & a_{k-1} & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0
\end{array} \\
\text{Polar Representation} & \quad +1 & +1 & -1 & -1 & +1 & -1 & -1 \\
\text{of} & \quad a_k & \text{(volts)} & 2 & 2 & 0 & -2 & 0 & 0 & -2 \\
\text{Decision} \{^\wedge b_k\} & \quad 0 & 0 & 1 & 0 & 1 & 1 & 0
\end{align*}
\]

Notice that in the absence of noise, \(\hat{b}_k = b_k\).
Modified Duobinary Technique

The modified duobinary signaling scheme can be shown as follows:

\[
\{b_k\} \xrightarrow{\text{Modulo-2 add}} \{a_k\} \xrightarrow{\text{Delay } 2T_b} \{c_k\}
\]

\[H(f)\]

\[t=KT_b\]

\[c_k = a_k - a_{k-2}, \quad a_k = b_k + a_{k-2}\]

Here, again, we find that a three-level signal is generated. If \(q_k = \pm 1\) V, then we set:

\[c_k = \begin{cases} 
0 & q_{k-2} = q_{k-2} = 0 \text{ Volts, } b_k = 0. \\
+2 & q_{k-2} - q_{k-2} = +2 \text{ Volts, } b_k = 1 \\
-2 & q_{k-2} - q_{k-2} = -2 \text{ Volts, } b_k = 1 
\end{cases}\]
\[ H(f) = H_c(f) \left[ 1 - e^{-j4\pi fT_b} \right] \]

\[ = 2j H_c(f) \sin(2\pi fT_b) e^{-j2\pi fT_b} \]

\[ \text{if } |f| \leq \frac{1}{2T_b} \]

\[ = 0 \text{ elsewhere.} \]

A useful feature of this signaling is that its output has no dc component. This is important since, in practice, many communications channels can't transmit a dc component!