

Synchronization methods Based on properties of wide-sense

cyclostationary random processes:

1. Carrier Recovery Circuits

wish to apply cyclostationarity to establish a local reference which is phase coherent with the phase of the received carrier.

Consider a modulated signal of the form

$$s(t) = \text{Re} \left\{ d(t) e^{j\theta} e^{j\omega_0 t} \right\}$$

data related

where θ and ω_0 are constants. For example, for a

BPSK
$$d(t) = \sum_{-\infty}^{\infty} a_k P(t - kT_b - \Delta), \quad a_k \in \{-1, 1\}$$

uniformly-dist.

In general, $d(t)$ will be assumed to be a complex-valued stationary process.

see previous slide.

$$\therefore R_{ss}(t, t+\tau) = E \left\{ \text{Re} \left[d(t) e^{j(\omega_0 t + \theta)} \right] \text{Re} \left[d(t+\tau) e^{j(\omega_0(t+\tau) + \theta)} \right] \right\}$$

← $\text{Re}\{z\} = \frac{z+z^*}{2}$

$$= E \left\{ \left[\frac{1}{2} d(t) e^{j(\omega_0 t + \theta)} + \frac{1}{2} d^*(t) e^{-j(\omega_0 t + \theta)} \right] \right.$$

$$\left. \left[\frac{1}{2} d(t+\tau) e^{j(\omega_0(t+\tau) + \theta)} + \frac{1}{2} d^*(t+\tau) e^{-j(\omega_0(t+\tau) + \theta)} \right] \right\}$$

$$= \frac{1}{4} E \left\{ d(t) d(t+\tau) e^{j2(\omega_0 t + \theta)} \cdot e^{j\omega_0 \tau} \right\}$$

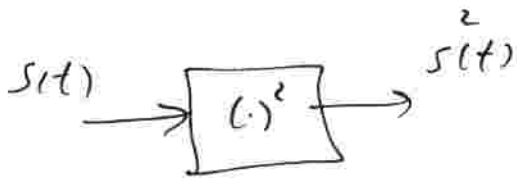
$$+ \frac{1}{4} E \left\{ d(t) d^*(t+\tau) e^{-j\omega_0 \tau} \right\} + \frac{1}{4} E \left\{ d^*(t) d(t+\tau) e^{j\omega_0 \tau} \right\}$$

$$+ \frac{1}{4} E \left\{ d^*(t) d^*(t+\tau) e^{-j2(\omega_0 t + \theta)} \cdot e^{-j\omega_0 \tau} \right\}$$

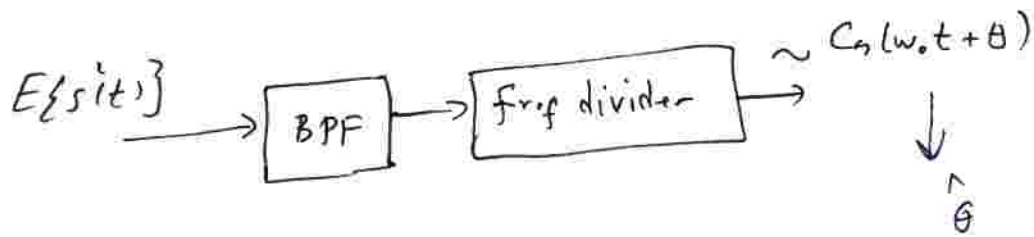
Aside: $\frac{1}{2}(x + x^*) = \text{Re}\{x\}$

$$\therefore R_{ss}(t, t+\tau) = \frac{1}{2} \text{Re} \left\{ R_{dd}(\tau) e^{j(2\omega_0 t + \omega_0 \tau + 2\theta)} \right\}$$

$$+ \frac{1}{2} \text{Re} \left\{ R_{dd^*}(\tau) e^{-j\omega_0 \tau} \right\}$$



$$\therefore E\{s^2(t)\} = R_{ss}(t, t) = \frac{1}{2} R_{dd^*}^{(0)} + \frac{1}{2} \operatorname{Re} \left\{ R_{dd}^{(0)} e^{j2(\omega_c t + \theta)} \right\}$$

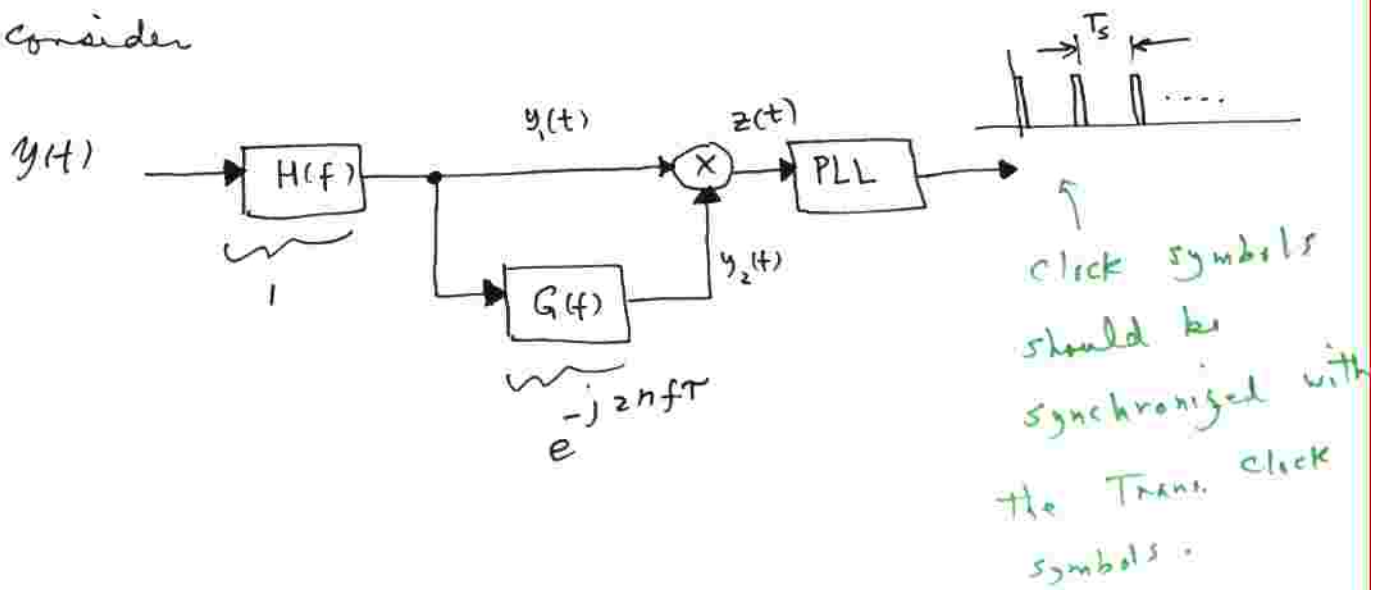


spectral
component at
twice the carrier
freq with
phase that is
twice that of
the antenna phase

For noise consideration, see PP 316-319.

Delay and multiply circuits for symbol clock estimation

consider



this problem is in fact timing epoch estimation.

Let $G(f) = e^{-j2\pi f\tau}$ ← pure delay

$$\therefore y_1(t) = s(t) + n(t)$$

$$y_2(t) = s(t-\tau) + n(t-\tau)$$

$$E\{z(t)\} = E\{y_1(t)y_2(t)\} = E\{s(t)s(t-\tau)\} + R_n(\tau)$$

$$\text{But } E\{s(t)s(t-\tau)\} = R_{ss}(t, t-\tau)$$

$$= \frac{1}{2} \text{Re} \left[R_{dd}^*(-\tau) e^{-j\omega_0\tau} \right] + \frac{1}{2} \text{Re} \left[R_{dd}^{(-\tau)} e^{j(2\omega_0 t - \omega_0\tau + 2\theta)} \right]$$

Recall that

$$s(t) = \text{Re} \left\{ d(t) e^{j\theta} e^{j\omega_0 t} \right\}$$

For QPSK, for example,

$$d(t) = d_r(t) + j d_i(t)$$

where

$$d_r(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{\infty} a_k P(t - kT_s)$$

$$d_i(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{\infty} b_k P(t - kT_s)$$

in which $\{a_n\}$ and $\{b_n\}$ are independent iid sequences

indep identically distributed

↓

iid sequences

$$\begin{aligned} \therefore R_{dd}(\tau) &= E \{ d(t) d(t-\tau) \} = E \left\{ [d_r(t) + j d_i(t)] [d_r(t-\tau) + j d_i(t-\tau)] \right\} \\ &= E \left\{ d_r(t) d_r(t-\tau) \right\} + E \left\{ j d_r(t) d_i(t-\tau) \right\} + E \left\{ j d_i(t) d_r(t-\tau) \right\} \\ &\quad - E \left\{ d_i(t) d_i(t-\tau) \right\} \end{aligned}$$

a_k and b_k are indep.

This has value only when $k=l \Rightarrow l-l=0$

$$\begin{aligned} &= \frac{A^2}{2} \sum_k \sum_l E \{ a_k a_l \} P(t - kT_s) P(t - \tau - lT_s) \\ &\quad - \frac{A^2}{2} \sum_k \sum_l E \{ b_k b_l \} P(t - kT_s) P(t - \tau - lT_s) = 0. \end{aligned}$$

indep. iid

$$\therefore R_{dd}(-\tau) \equiv 0.$$

Now consider

$$E\{d(t)d^*(t-\tau)\}$$

$$R_{dd^*}(-\tau) = E\{d_r(t)d_r(t-\tau)\} + E\{d_i(t)d_i(t-\tau)\}$$

$$= \frac{A^2}{2} \sum_k \sum_l E\{a_k a_l\} P(t-kT_s) P(t-\tau-lT_s)$$

$$+ \frac{A^2}{2} \sum_k \sum_l E\{b_k b_l\} P(t-kT_s) P(t-\tau-lT_s)$$

$$= \frac{A^2}{2} \sum_k \sum_l E\{a_k a_l + b_k b_l\} P(t-kT_s) P(t-\tau-lT_s)$$

$$\text{But } E\{a_k a_l + b_k b_l\} = \underbrace{E\{a_k a_l\}}_{\delta_{kl}} + \underbrace{E\{b_k b_l\}}_{\delta_{kl}} = 2\delta_{kl}$$

$$\therefore R_{dd^*}(-\tau) = A^2 \sum_{k=-\infty}^{\infty} P(t-kT_s) P(t-\tau-kT_s)$$

and

$$R_{ss}(t, t-\tau) = \frac{A^2}{2} \sum_{k=-\infty}^{\infty} P(t-kT_s) P(t-\tau-kT_s) \cos \omega_0 \tau$$

Recall that

$$R_{ss}(t, t-\tau) = \frac{1}{2} \operatorname{Re} \left\{ R_{dd^*}(-\tau) e^{-j\omega_0 \tau} \right\}$$

Note :

1. $R_{SS}(t, t-T)$ is max if $\cos \omega_0 T = 1$

$$\text{or } \omega_0 T = 2\pi m, \quad m = 0, 1, 2, \dots$$

as a function of time

2. $R_{SS}(t, t-T)$ is periodic in T_s .

Hence R_{SS} is cyclostationary.

Letting $\cos \omega_0 T = 1$, we can represent

$R_{SS}(t, t-T)$ in terms of a Fourier series:

$$\left. R_{SS}(t, t-T) \right|_{\cos \omega_0 T = 1} = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi t/T_s}$$

a time function
for T fixed!
where

$$C_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \left. R_{SS}(t, t-T) \right|_{\cos \omega_0 T = 1} e^{-jn2\pi t/T_s} dt$$

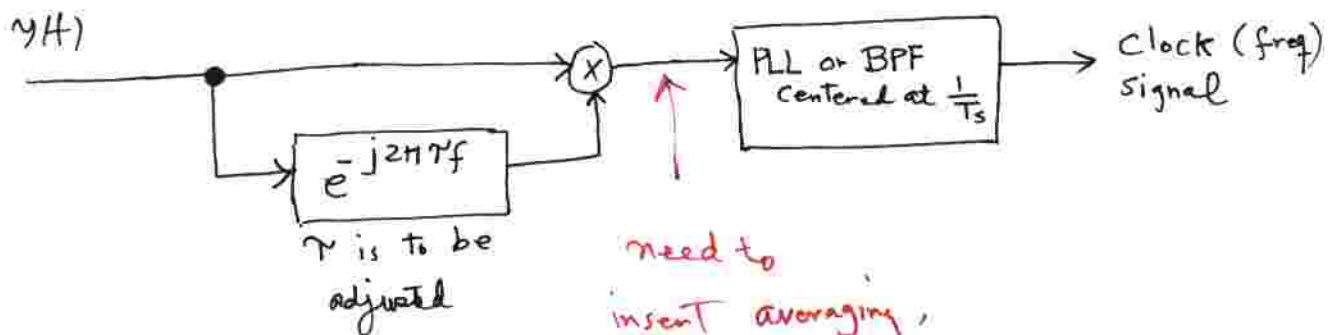
$$= \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \frac{A^2}{2} \sum_{k=-\infty}^{\infty} P(t-kT_s) P(t-T-kT_s) e^{-jn2\pi t/T_s} dt$$

$$= \frac{A^2}{2T_s} \int_{-\infty}^{\infty} P(t) P(t-T) e^{-jn2\pi t/T_s} dt$$

the component at the clock frequency is the 1st harmonic and has the power $|C_1|^2$.

$$\therefore \text{If } \omega_0 \tau = 2m\pi, \quad m = \text{integer}$$

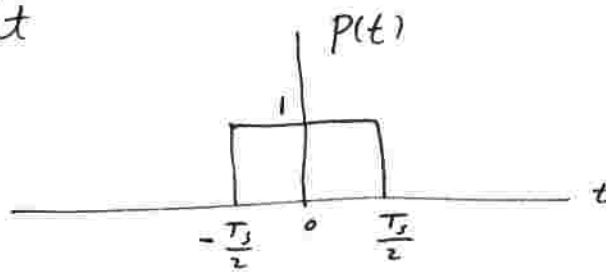
then we need to adjust τ to maximize $|C_1|^2$.



however, PLL does averaging but BPF does not!

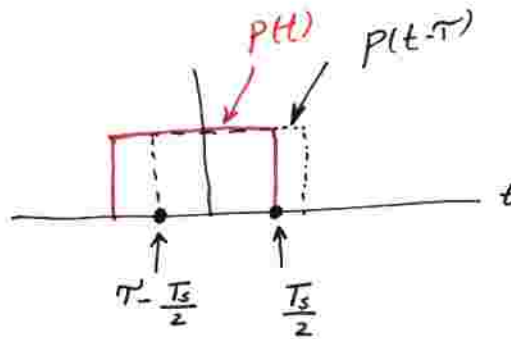
For noise considerations, see ^{the} link!

Exa. Let

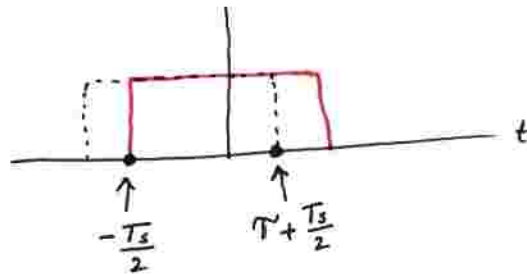


$$\therefore C_n = \frac{1}{2T_s} \int_{-\infty}^{\infty} p(t) p(t-\tau) e^{-j2\pi n t / T_s} dt$$

If $0 \leq \tau \leq \frac{T_s}{2}$



If $-\frac{T_s}{2} < \tau < 0$



$$C_n = (-1)^n \frac{|\tau|}{2T_s} \text{sinc}\left(\frac{n\tau}{T_s}\right) \exp\left(\frac{-j\pi n |\tau|}{T_s}\right); \quad |\tau| \leq \frac{T_s}{2}$$

$$|C_1|^2 = \left(\frac{\tau}{2T_s}\right)^2 \text{sinc}^2\left(\frac{\tau}{T_s}\right)$$

$$|C_1|_{\max}^2 = 0.0253 \text{ at } \tau = \frac{T_s}{2}$$