Synchronization methods Based on properties of Wide-sense cyclostationary random processes:

1. Carrier Recovery Circuits

wish to apply cyclostationarity to establish a local reference which is phase coherent with the phase of the received carrier.

Consider a modulated signal of the form

$$s(t) = \mathrm{Re}\left\{ d(t)\, e^{j\theta} e^{j\omega_0 t} \right\}.$$

data related

where θ and ω_0 are constants. For example, for a BPSK

$$d(t) = \sum_{-\infty}^{\infty} a_k\, P(t - kT_b - \Delta) \quad , \quad a_k \in \{-1, 1\}$$

uniformly-dist.

In general, $d(t)$ will be assumed to be a complex-valued stationary process.

See previous aside.

$$\therefore \quad R_{ss}(t, t+\tau) = E\left\{ Re\left[d(t) e^{j(w_o t+\theta)} \right] Re\left[d(t+\tau) e^{j(w_o(t+\tau)+\theta)} \right] \right\}$$

$Re\{z\} = \frac{z+z^*}{2}$

$$= E\left\{ \left[\frac{1}{2} d(t) e^{j(w_o t+\theta)} + \frac{1}{2} d^*(t) e^{-j(w_o t+\theta)} \right] \left[\frac{1}{2} d(t+\tau) e^{j(w_o(t+\tau)+\theta)} + \frac{1}{2} d^*(t+\tau) e^{-j(w_o(t+\tau)+\theta)} \right] \right\}$$

$$= \frac{1}{4} E\left\{ d(t)\, d(t+\tau)\, e^{2j(w_o t+\theta)} \cdot e^{j w_o \tau} \right\}$$

$$+ \frac{1}{4} E\left\{ d(t)\, d^*(t+\tau)\, e^{-j w_o \tau} \right\} + \frac{1}{4} E\left\{ d^*(t)\, d(t+\tau)\, e^{j w_o \tau} \right\}$$

$$+ \frac{1}{4} E\left\{ d^*(t)\, d^*(t+\tau)\, e^{-j2(w_o t+\theta)} \cdot e^{-j w_o \tau} \right\}.$$

$\boxed{\underline{\text{Aside}}: \quad \frac{1}{2}(x + x^*) = Re\{x\}}$

$$\therefore \quad R_{ss}(t, t+\tau) = \frac{1}{2} Re\left\{ R_{dd}(\tau)\, e^{j(2w_o t + w_o \tau + 2\theta)} \right\}$$

$$+ \frac{1}{2} Re\left\{ R_{dd^*}(\tau)\, e^{-j w_o \tau} \right\}$$

$$s(t) \longrightarrow \boxed{(\cdot)^2} \longrightarrow s^2(t)$$

$$\therefore \quad E\{s^2(t)\} = R_{ss}(t,t) = \frac{1}{2} R_{dd^*}(0) + \frac{1}{2} Re\left\{ R_{dd}(0)\, e^{j2(w_0 t+\theta)} \right\}$$

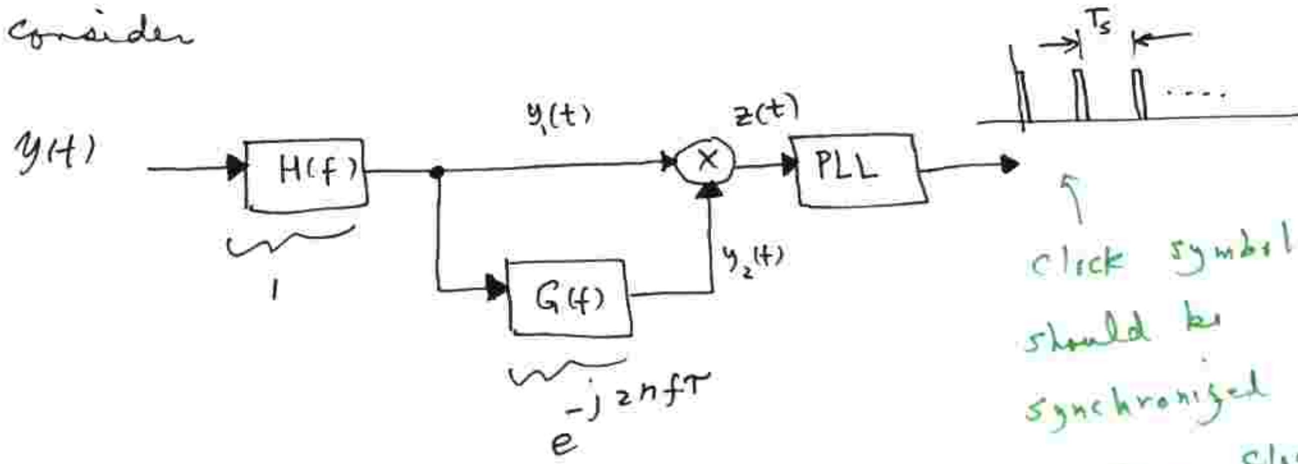
spectral Component at twice the Carrier freq with Phase that is twice that of the unknown phase

$$E\{s^2(t)\} \longrightarrow \boxed{BPF} \longrightarrow \boxed{\text{freq divider}} \longrightarrow \sim \cos(w_0 t+\theta) \longrightarrow \hat{\theta}$$

For noise consideration, see pp 316 - 319.

Delay and multiply circuits for symbol clock Estimation

Consider



clock symbols should be synchronized with the Trans. clock symbols.

this problem is in fact timing epoch estimation.

Let $\quad G(f) = e^{-j2\pi f T} \quad \leftarrow$ Pure delay

$$\therefore \quad y_1(t) = s(t) + n(t)$$

$$y_2(t) = s(t-\tau) + n(t-\tau)$$

$$E\{z(t)\} = E\{y_1(t)\, y_2(t)\} = E\{s(t)s(t-\tau)\} + R_n(\tau)$$

But $\quad E\{s(t)s(t-\tau)\} = R_{ss}(t, t-\tau)$

$$= \frac{1}{2} Re\left[R_{dd^*}(-\tau)\, e^{-jw_0\tau} \right] + \frac{1}{2} Re\left[R_{dd}(-\tau)\, e^{j(2w_0 t - w_0\tau + 2\theta}\right.$$

Recall that

$$s(t) = Re\{ d(t)\, e^{j\theta} e^{j\omega_0 t} \}$$

For QPSK, for example,

$$d(t) = d_r(t) + j\, d_i(t)$$

Where

$$d_r(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{\infty} a_k P(t - kT_s)$$

$$d_i(t) = \frac{A}{\sqrt{2}} \sum_{k=-\infty}^{\infty} b_k P(t - kT_s).$$

indep identically distributed

in which $\{a_n\}$ (± 1) and $\{b_n\}$ (± 1) are independent iid sequences

$$\therefore \quad R_{dd}(-\tau) = E\{ d(t)\, d(t-\tau) \} = E\left\{ [d_r(t) + j\, d_i(t)][d_r(t-\tau) + j\, d_i(t-\tau)] \right\}$$

a_k and b_k are indep.

$$= E\{ d_r(t)\, d_r(t-\tau) \} + \underbrace{\cancel{E\{ j\, d_r(t)\, d_i(t-\tau) \}}}_{0} + \underbrace{\cancel{E\{ j\, d_i(t)\, d_r(t-\tau) \}}}_{0}$$

$$- E\{ d_i(t)\, d_i(t-\tau) \}$$

This has value only when $k = l \Rightarrow 1 - 1 = 0$

$$= \frac{A^2}{2} \sum_k \sum_l E\{ a_k a_l \} P(t - kT_s) P(t - \tau - lT_s)$$

$$- \frac{A^2}{2} \sum_k \sum_l E\{ b_k b_l \} P(t - kT_s) P(t - \tau - lT_s) = 0.$$

$$\therefore \quad R_{dd}(-\tau) \equiv 0.$$

Now consider $E\{d(t)\, d^*(t-\tau)\}$

$$R_{dd^*}(-\tau) = E\{d_r(t)\, d_r(t-\tau)\} + E\{d_i(t)\, d_i(t-\tau)\}$$

$$= \frac{A^2}{2}\sum_k\sum_\ell E\{a_k a_\ell\}\, P(t-kT_s)\, P(t-\tau-\ell T_s)$$

$$+ \frac{A^2}{2}\sum_k\sum_\ell E\{b_k b_\ell\}\, P(t-kT_s)\, P(t-\tau-\ell T_s)$$

$$= \frac{A^2}{2}\sum_k\sum_\ell E\{a_k a_\ell + b_k b_\ell\}\, P(t-kT_s)\, P(t-\tau-\ell T_s)$$

$$\text{But} \quad E\{a_k a_\ell + b_k b_\ell\} = \underbrace{E\{a_k a_\ell\}}_{\delta_{k\ell}} + \underbrace{E\{b_k b_\ell\}}_{\delta_{k\ell}} = 2\delta_{k\ell}$$

$$\therefore \quad R_{dd^*}(-\tau) = A^2 \sum_{k=-\infty}^{\infty} P(t-kT_s)\, P(t-\tau-kT_s)$$

and

$$R_{ss}(t, t-\tau) = \frac{A^2}{2}\sum_{k=-\infty}^{\infty} P(t-kT_s)\, P(t-\tau-kT_s)\cos\omega_0\tau .$$

Recall that $R_{ss}(t,t-\tau) = \frac{1}{2}\mathrm{Re}\{R_{dd^*}(-\tau)\, e^{-j\omega_0\tau}\}$

Note :

1. $R_{ss}(t, t-\tau)$ is max if $\cos w_0\tau = 1$

or $w_0 \tau = 2\pi m$, $m = 0, 1, 2, \ldots$

2. $R_{ss}(t, t-\tau)$ as a fcn of time is periodic in T_s .

thus R_{ss} is Cyclostationary.

$\therefore$ Letting $\cos w_0 \tau = 1$, we can represent $R_{ss}(t, t-\tau)$ in terms of a Fourier series :

$$\underbrace{R_{ss}(t, t-\tau)}_{\substack{\text{a time fcn}\\ \text{for } \tau \text{ fixed!}}}\Big|_{\cos w_0\tau = 1} = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi t/T_s}$$

where

$$C_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} R_{ss}(t, t-\tau)\Big|_{\cos w_0\tau=1} e^{-jn2\pi t/T_s} dt$$

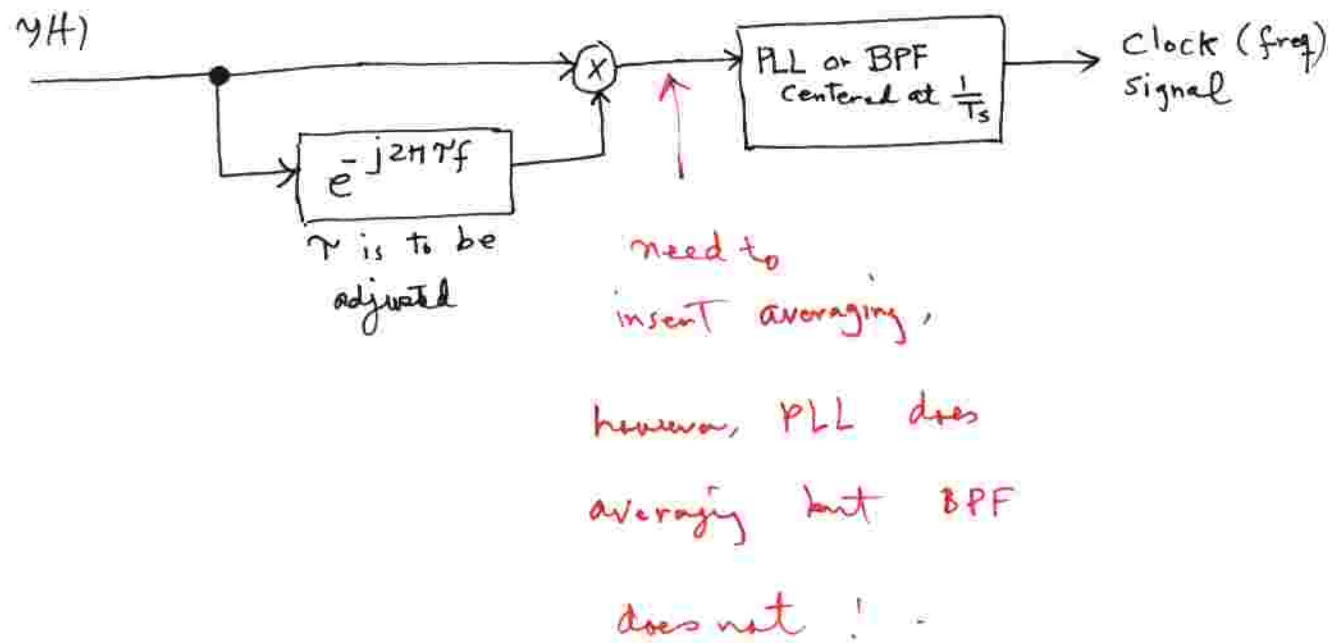
$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \frac{A^2}{2}\sum_{k=-\infty}^{\infty} P(t-kT_s)P(t-\tau-kT_s)\, e^{-jn2\pi t/T_s} dt .$$

$$= \frac{A^2}{2T_s}\int_{-\infty}^{\infty} P(t)P(t-\tau)\, e^{-jn2\pi t/T_s} dt .$$

the component at the clock frequency is the 1ST harmonic and has the power $|C_1|^2$.

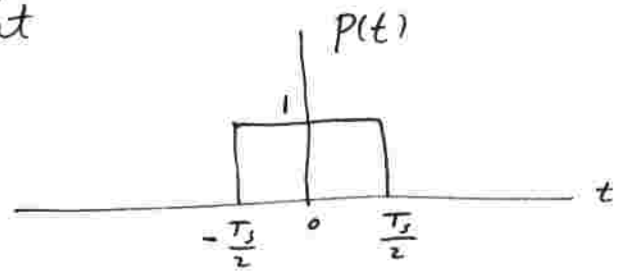
$\therefore$ If $\omega_0 \tau = 2m\pi$, m = integer

then we need to adjust τ to maximize $|C_1|^2$.



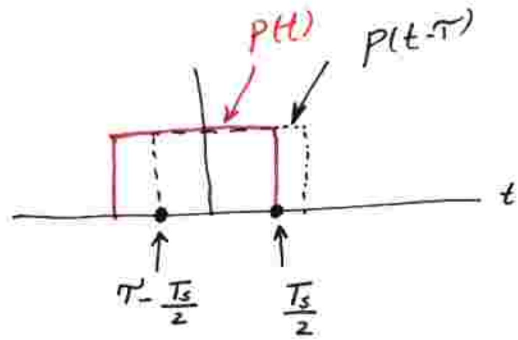
For noise considerations, see the book!

Exa. Let

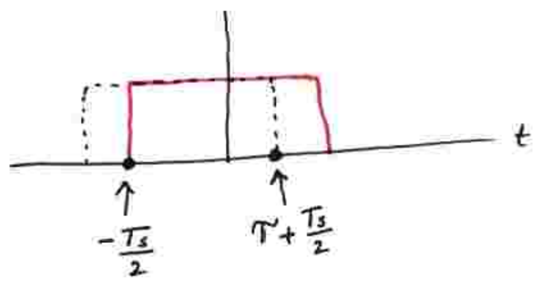


$$\therefore \quad C_n = \frac{1}{2T_s}\int_{-\infty}^{\infty} p(t)\, p(t-\tau)\, e^{-j2\pi n t/T_s}\, dt$$

If $0 \le \tau \le \frac{T_s}{2}$



If $-\frac{T_s}{2} \le \tau < 0$



$\therefore$

$$C_n = (-1)^n \frac{|\tau|}{2T_s}\, \mathrm{sinc}\left(\frac{n\tau}{T_s}\right) \exp\left(\frac{-j\pi n|\tau|}{T_s}\right); \qquad |\tau| \le \frac{T_s}{2}$$

$$|C_1|^2 = \left(\frac{\tau}{2T_s}\right)^2 \mathrm{sinc}^2\left(\frac{\tau}{T_s}\right)$$

$$|C_1|^2_{max} = .0253 \quad \text{at} \quad \tau = \frac{T_s}{2}.$$