

Another look at LRT (MAP)

$$\Pr(H_1 | Y \leq y \leq Y + dy) \underset{H_0}{\overset{H_1}{\geq}} \Pr(H_0 | Y \leq y \leq Y + dy)$$

$$\therefore \frac{\Pr(Y \leq y \leq Y + dy | H_1) P(H_1)}{\Pr(Y \leq y \leq Y + dy)} \underset{H_0}{\overset{H_1}{\geq}} \frac{\Pr(Y \leq y \leq Y + dy | H_0) P(H_0)}{\Pr(Y \leq y \leq Y + dy)}$$

But $\Pr(Y \leq y \leq Y + dy | H_1) = P_{y|H_1}(Y|H_1) dy$

$$\Pr(Y \leq y \leq Y + dy) = P_y(Y) dy$$

$$\Pr(Y \leq y \leq Y + dy | H_0) = P_{y|H_0}(Y|H_0) dy$$

$$P_{y|H_1}(Y|H_1) P_1 \underset{H_0}{\overset{H_1}{\geq}} P_{y|H_0}(Y|H_0) P_0$$

$$\therefore \frac{P_{y|H_1}(Y|H_1) dy P_1}{\cancel{P_y(Y) dy}} \underset{H_0}{\overset{H_1}{\geq}} \frac{P_{y|H_0}(Y|H_0) P_0}{\cancel{P_y(Y) dy}}$$

$$\therefore \frac{P_{y|H_1}(Y|H_1)}{P_{y|H_0}(Y|H_0)} \underset{H_0}{\overset{H_1}{\geq}} \frac{P_0}{P_1}$$

$\Lambda(Y)$

← this is Bayes test
with $c_{00} = c_{11} = 0$
 $c_{10} = c_{01}$

Neyman-Pearson Criterion (Radar Pb $\frac{m}{s}$).

In many physical situations it is difficult to assign realistic costs or a priori probs. A simple procedure to bypass this is to work with P_F and P_D .

In general, we like to make

$$P_F \downarrow \quad P_D \uparrow$$

In this method we constrain $P_F = \alpha' \leq \alpha$

and design a test to maximize P_D (or min P_M).

the solⁿ is obtained by using Lagrange multipliers.

$$P_r(D_0|H_1) \quad P_r(D_1|H_0)$$

$$F = \cancel{P_M} + \lambda (\cancel{P_F} - \alpha').$$

$$= \int_{z_1} P_{r|H_1}(\underline{R}|H_1) d\underline{R} + \lambda \left[\int_{z_1} P_{r|H_0}(\underline{R}|H_0) d\underline{R} - \alpha' \right].$$

$\alpha' < 1$ because equal to P_F

$$= \lambda(1 - \alpha') + \int_{z_0} [P_{r|H_1}(\underline{R}|H_1) - \lambda P_{r|H_0}(\underline{R}|H_0)] d\underline{R}$$

for $\lambda > 0$, F is minimum if the integrand is < 0 .

$\therefore R \in Z_0$ iff:

$$P_{\underline{r}|H_1}(R|H_1) - \lambda P_{\underline{r}|H_0}(R|H_0) < 0$$

$$\therefore \Lambda(R) = \frac{P_{\underline{r}|H_1}(R|H_1)}{P_{\underline{r}|H_0}(R|H_0)} \underset{H_0}{\underset{H_1}{\gtrless}} \lambda \quad \leftarrow \text{LRT}$$

the threshold λ can be obtained as follows:



choose λ so that $P_F = \alpha'$ fixed ratio.

But

$$P_F = \int_{\lambda}^{\infty} P(\Lambda|H_0) d\Lambda = \alpha'$$

means H_0 is sent
 But $\Lambda > \lambda$
 \therefore false alarm!!

Aside :

It may not be necessary to calculate $P_{\underline{\Lambda}|H_0}$ in Neyman-Pearson criterion. We may be able to find a sufficient statistic.

$$\underline{\Lambda}(\underline{R}) = \frac{P_{\underline{R}|H_1}(\underline{R}|H_1)}{P_{\underline{R}|H_0}(\underline{R}|H_0)}$$

by selecting the variables properly

$$\underline{\Lambda}(L, Y) = \frac{P_{L, Y|H_1}(L, Y|H_1)}{P_{L, Y|H_0}(L, Y|H_0)} =$$

$$= \frac{P_{Y|L, H_1}(Y|L, H_1) \underset{\downarrow}{P(L|H_1)}}{P_{Y|L, H_0}(Y|L, H_0) \underset{\downarrow}{P(L|H_0)}} = \frac{\underset{\downarrow}{P(L|H_1)}}{\underset{\downarrow}{P(L|H_0)}}$$

2. Consider a single measurement $r = s + n$, where $n \sim (0, 2)$ and s is a constant equal to either 0 or 1. Determine an optimum decision rule to choose between the hypotheses

$$H_0: s=0$$

$$H_1: s=1$$

Using a Neyman-Pearson test with $P_F = 0.1$.

$$H_1: r = 1 + n$$

$$H_0: r = n$$

$$P_{r|H_1}(R|H_1) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(R-1)^2}{2\sigma^2}\right)$$

$$P_{r|H_0}(R|H_0) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{R^2}{2\sigma^2}\right)$$

$$\Lambda(R) = \exp\left(-\left[\frac{(R-1)^2}{4} - \frac{R^2}{4}\right]\right) \underset{H_0}{\underset{H_1}{\gtrless}} \lambda_0$$

$$e^{\frac{R}{2} - \frac{1}{4}} \underset{H_0}{\underset{H_1}{\gtrless}} \lambda_0 \Rightarrow \frac{R}{2} - \frac{1}{4} \underset{H_0}{\underset{H_1}{\gtrless}} \ln \lambda_0$$

$$\text{or } R \underset{H_0}{\underset{H_1}{\gtrless}} 2(\ln \lambda_0 + \frac{1}{4}) = \gamma$$

$$P_F = \int_{\gamma}^{\infty} P_{r|H_0}(R|H_0) dR = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}(2)} \exp\left(-\frac{R^2}{4}\right) dR = 0.1$$

$$\text{Let } R' = \frac{R}{\sqrt{2}} \Rightarrow P_F = \int_{\frac{\gamma}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{R'^2}{2}\right) dR' = 0.1$$

$$\therefore R \underset{H_0}{\underset{H_1}{\gtrless}} 1.83$$

$$Q\left(\frac{\gamma}{\sqrt{2}}\right) = 0.1 \Rightarrow \frac{\gamma}{\sqrt{2}} \approx 1.3$$

$$\gamma \approx 1.83$$

Performance: Receiver operating characteristic (ROC)

To complete our discussion of binary Pbm we consider ROC.

Exa 1 (Revisited)

$$H_1: r_i = m + n_i$$

$$H_0: r_i = n_i$$

$n_i, n_j, i \neq j$ are indep

$$P_{n_i}(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-X^2/2\sigma^2)$$

$$l = \frac{1}{\sqrt{N}\sigma} \sum_{i=1}^N R_i \begin{matrix} H_1 \\ \geq \\ H_0 \end{matrix} \left(\frac{\sigma^2}{m} \ln \eta + \frac{Nm}{2} \right) \frac{1}{\sqrt{N}\sigma} = \frac{\sigma}{\sqrt{Nm}} \ln \eta + \frac{\sqrt{N}m}{2\sigma}$$

$$l = \frac{R_1}{\sqrt{N}\sigma} + \frac{R_2}{\sqrt{N}\sigma} + \dots + \frac{R_N}{\sqrt{N}\sigma}$$

$$\sigma_l^2 = \frac{1}{N\sigma^2} E\{R_1^2\} + \dots + \frac{1}{N\sigma^2} E\{R_N^2\}$$

(R_i 's are indep)

Under H_0 , R_i has zero mean and variance of σ^2

$$\therefore \sigma_l^2 = \frac{1}{N\sigma^2} \sigma^2 + \dots + \frac{1}{N\sigma^2} \sigma^2 = 1$$

$$\therefore l_{H_0} = \mathcal{N}(0, 1)$$

Under H_1 , R_i has mean of m and variance of σ^2

$$\sigma_l^2 \Rightarrow E\{l\} = \frac{m}{\sqrt{N}\sigma} + \dots + \frac{m}{\sqrt{N}\sigma} = \frac{Nm}{\sqrt{N}\sigma} = \frac{\sqrt{N}m}{\sigma}$$

$$\therefore \sigma_l^2 = \frac{1}{N\sigma^2} \sigma^2 + \dots + \frac{1}{N\sigma^2} \sigma^2 = 1$$

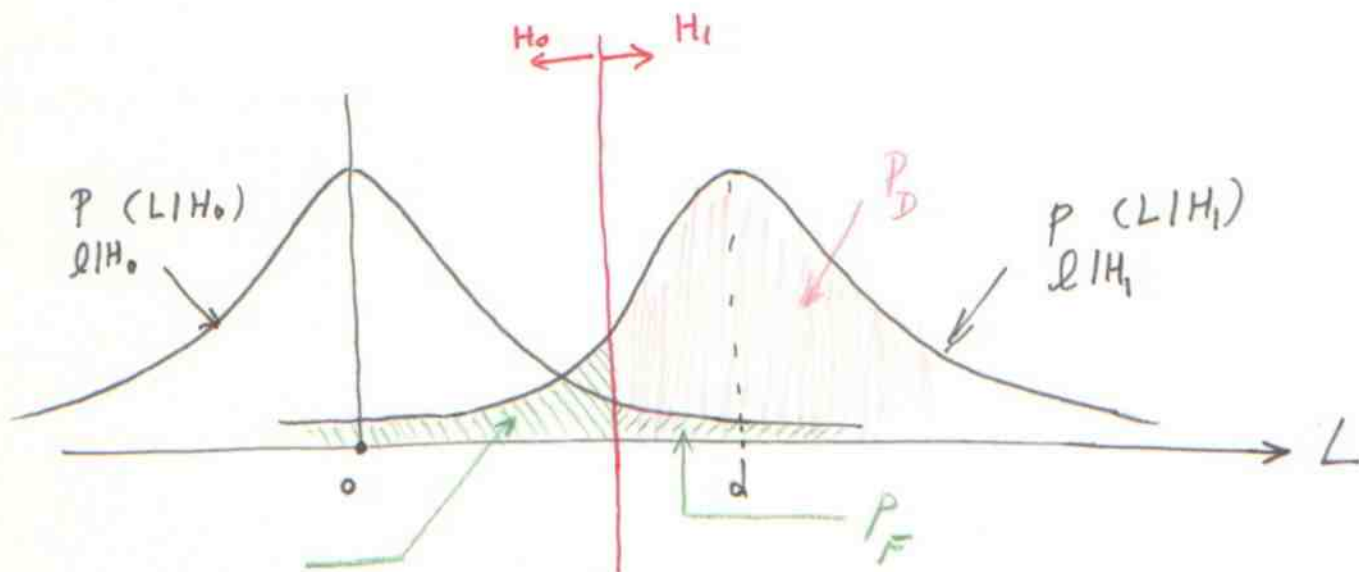
$$\therefore l_{H_1} = \mathcal{N}\left(\frac{\sqrt{N}m}{\sigma}, 1\right)$$

$$P_{l|H_0} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{l^2}{2}\right)$$

$$P_{l|H_1} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left[l - \frac{m\sqrt{N}}{\sigma}\right]^2}{2}\right)$$

note:

$$d^2 = \frac{Nm^2}{\sigma^2} = \text{SNR}$$



$$\frac{\sigma}{\sqrt{N}m} \ln \eta + \frac{\sqrt{N}m}{2\sigma^2}$$

$\underbrace{\hspace{10em}}_{\frac{\ln \eta}{d} + \frac{d}{2}}$

$$P_F = \int_{\frac{\ln \eta}{d} + \frac{d}{2}}^{\infty} P(L|H_0) dL$$

$$P_M = \int_{-\infty}^{\frac{\ln \eta}{d} + \frac{d}{2}} P(L|H_1) dL$$

note :

$$\text{erf}_* (x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-x^2/2} dx$$

$$\text{erfc}_* (x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2} dx$$

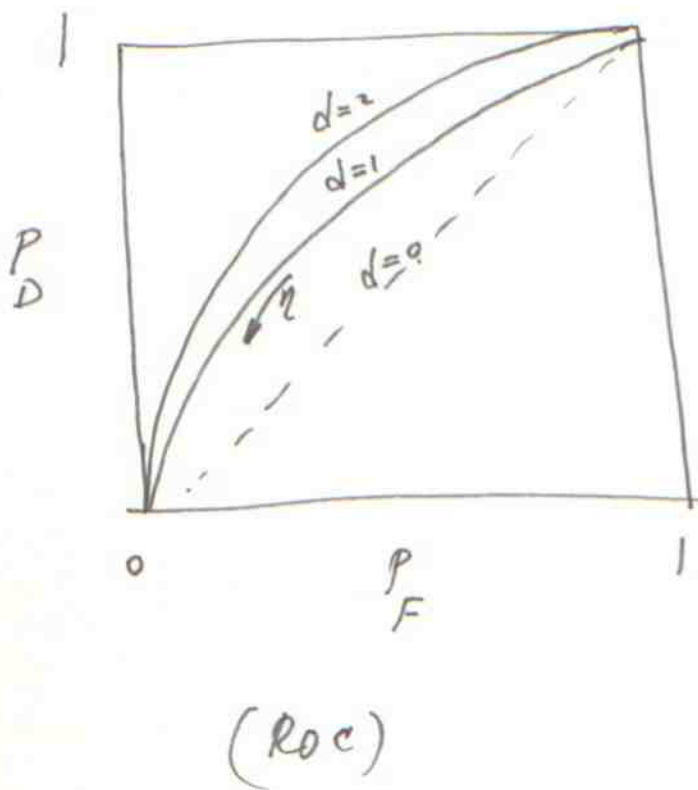
note that

$$P_F = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln \eta}{d} + \frac{d}{2}}^{\infty} e^{-y^2/2} dy = \operatorname{erfc}_* \left(\frac{\ln \eta}{d} + \frac{d}{2} \right).$$

$$P_D = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln \eta}{d} + \frac{d}{2}}^{\infty} \exp\left(-\frac{(x-d)^2}{2}\right) dx$$

let $y = x - d$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln \eta}{d} - \frac{d}{2}}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = \operatorname{erfc}_* \left(\frac{\ln \eta}{d} - \frac{d}{2} \right).$$



A special case of interest is

$$Pr(\epsilon) = P_0 P_F + P_1 \overset{1-P_D}{P_M}$$

the threshold for this criterion is

$$\gamma = \ln \eta = \ln P_0 - \ln(1 - P_0)$$

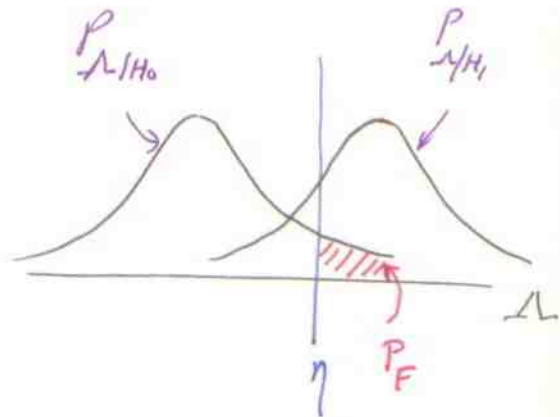
thus for specific d we can find P_D and P_F and thus $Pr(\epsilon)$ from ROC. ($P_D = 1 - P_M$).

Property 3 (ROC)

the slope of the ROC at a particular point is equal to the threshold η required to achieve P_D and P_F at that point.

$$P_D : P_D = \int_{\eta}^{\infty} P_{\Lambda|H_1}(\Lambda|H_1) d\Lambda$$

$$P_F = \int_{\eta}^{\infty} P_{\Lambda|H_0}(\Lambda|H_0) d\Lambda$$



$$\frac{dP_D/d\eta}{dP_F/d\eta} = \frac{-P_{\Lambda|H_1}(\eta|H_1)}{-P_{\Lambda|H_0}(\eta|H_0)} = \frac{dP_D}{dP_F} \quad (1)$$

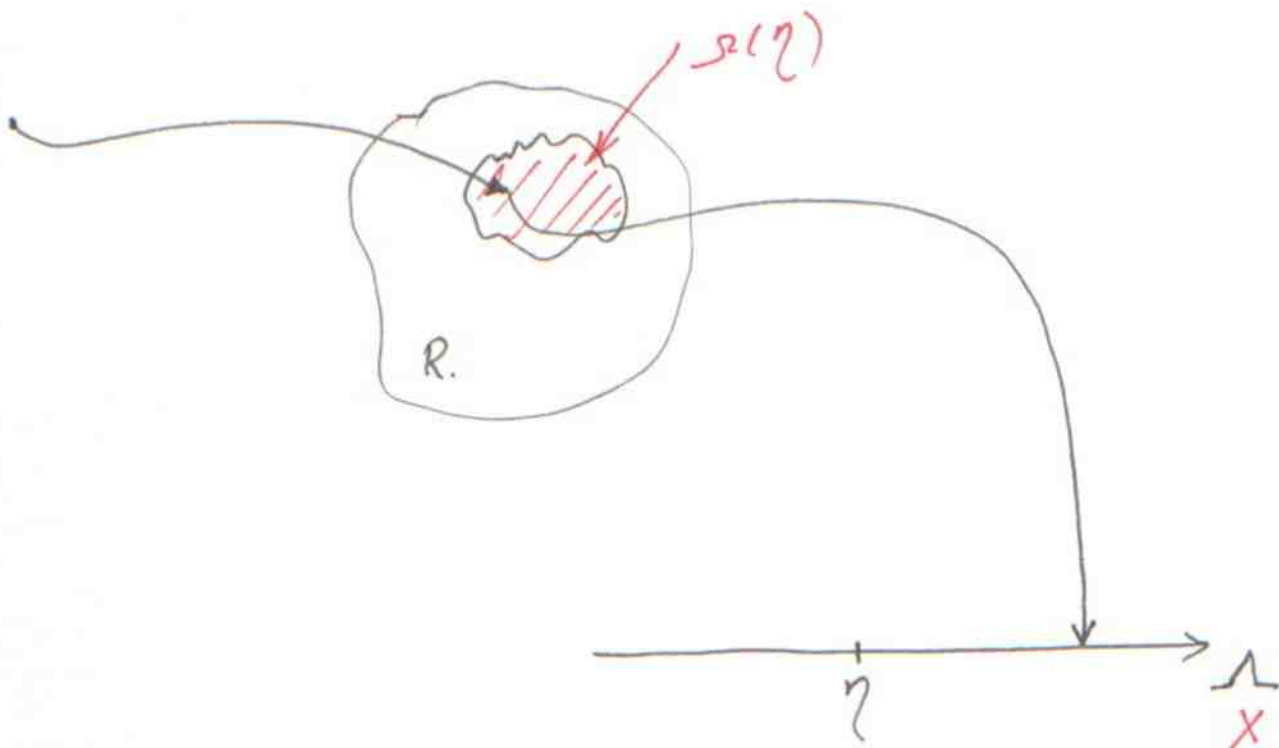
wish To show

$$\eta = \frac{P_{\Lambda|H_1}(\eta|H_1)}{P_{\Lambda|H_0}(\eta|H_0)}$$

let

$$\Omega(\eta) \triangleq \{ \underline{R} \mid \Lambda(\underline{R}) \geq \eta \}$$

Source.



$$\therefore \Omega(\eta) \triangleq \left\{ \underline{R} \mid \frac{P_{\underline{R}|H_1}(\underline{R}|H_1)}{P_{\underline{R}|H_0}(\underline{R}|H_0)} \geq \eta \right\}$$

then

$$P_D(\eta) = P_{\underline{R}} \{ \Lambda(\underline{R}) \geq \eta \mid H_1 \} = \int_{\Omega(\eta)} P_{\underline{R}|H_1}(\underline{R}|H_1) d\underline{R}$$

$$= \int_{\Omega(\eta)} \left[\frac{P_{\underline{R}|H_1}(\underline{R}|H_1)}{P_{\underline{R}|H_0}(\underline{R}|H_0)} \right] P_{\underline{R}|H_0}(\underline{R}|H_0) d\underline{R} = \int_{\Omega(\eta)} \Lambda(\underline{R}) P_{\underline{R}|H_0}(\underline{R}|H_0) d\underline{R}$$

But

$$P_D(\eta) = \int_{\Omega(\eta)} \mathcal{L}(\underline{R}) P_{\underline{R}|H_0}(\underline{R}|H_0) d\underline{R} = \int_{\eta}^{\infty} x P(x|H_0) \mathcal{L}|H_0 dx$$

$$\frac{dP_D(\eta)}{d\eta} = -\eta \frac{P(\eta|H_0)}{\mathcal{L}|H_0}$$

$$\therefore \eta = -\frac{1}{\frac{P(\eta|H_0)}{\mathcal{L}|H_0}} \left(\frac{dP_D}{d\eta} \right) = -\frac{1}{\frac{P(\eta|H_0)}{\mathcal{L}|H_0}} \left[-\frac{P(\eta|H_0)}{\mathcal{L}|H_0} \right] \quad (2)$$

(1) & (2) \Rightarrow

$$\eta = \frac{dP_D}{dP_F}$$