Another look at LRT (MAP)

$$Pr(H_1 \mid Y \le y \le Y + dY) \underset{H_0}{\overset{H_1}{\gtrless}} Pr(H_0 \mid Y \le y \le Y + dY)$$

$$\therefore \quad \frac{Pr(Y \le y \le Y + dY \mid H_1)\, Pr(H_1)}{Pr(Y \le y \le Y + dY)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{Pr(Y \le y \le Y + dY \mid H_0)\, Pr(H_0)}{Pr(Y \le y \le Y + dY)}$$

But $$Pr(Y \le y \le Y + dY \mid H_1) = P_{y|H_1}(Y \mid H_1)\, dY$$

$$Pr(Y \le y \le Y + dY) = P_y(Y)\, dY$$

$$Pr(Y \le y \le Y + dY \mid H_0) = P_{y|H_0}(Y \mid H_0)\, dY$$

$$P_{y|H_1}(Y|H_1)\, P_1 \underset{H_0}{\overset{H_1}{\gtrless}} P_{y|H_0}(Y|H_0)\, P_0$$

$$\therefore \quad \frac{P_{y|H_1}(Y \mid H_1)\, \cancel{dY}\, P_1}{\cancel{P_y(Y)\, dY}} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_{y|H_0}(Y \mid H_0)\, \cancel{dY}\, P_0}{\cancel{P_y(Y)\, dY}}$$

$$\therefore \quad \underbrace{\frac{P_{y|H_1}(Y \mid H_1)}{P_{y|H_0}(Y \mid H_0)}}_{\Lambda(Y)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P_0}{P_1}$$

← this is Bayes test with $C_{00} = C_{11} = 0$, $C_{10} = C_{01}$

Neyman-Pearson Criterion (Radar Pb $\frac{m}{s}$).

In many physical situations it is difficult to assign realistic costs or a priori probs. A simple procedure to bypass this is to work with P_F and P_D.

In general, we like to make

$$P_F \downarrow \qquad P_D \uparrow$$

In this method we constrain $P_F = \alpha' \leq \alpha$ and design a test to maximize P_D (or min P_M).

the sol^n is obtained by using Lagrange multipliers.

$$\qquad \Pr(D_0|H_1) \qquad \Pr(D_1|H_0)$$

$$F = P_M + \lambda (P_F - \alpha').$$

$$= \int_{Z_0} p_{\underline{r}|H_1}(\underline{R}|H_1)\, d\underline{R} + \lambda \left[\int_{\cancel{Z_1}\ Z - Z_0} p_{\underline{r}|H_0}(\underline{R}|H_0)\, d\underline{R} - \alpha' \right].$$

$\alpha' < 1$, if Z_0 because it's equal to P_F

$$= \lambda (1 - \alpha') + \int_{Z_0} \left[p_{\underline{r}|H_1}(\underline{R}|H_1) - \lambda\, p_{\underline{r}|H_0}(\underline{R}|H_0) \right] d\underline{R}.$$

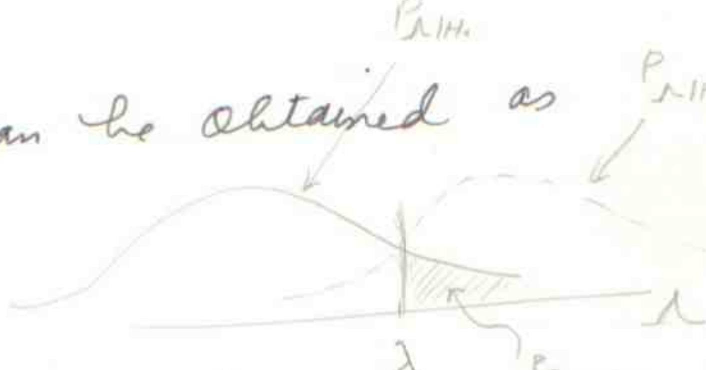
for $\lambda > 0$, F is minimum if the integrand is < 0.

$\therefore \quad R \in Z_0$ iff :

$$P_{\underline{r}|H_1}(\underline{R}|H_1) - \lambda P_{\underline{r}|H_0}(\underline{R}|H_0) < 0$$

$$\therefore \quad \Lambda(\underline{R}) = \frac{P_{\underline{r}|H_1}(\underline{R}|H_1)}{P_{\underline{r}|H_0}(\underline{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda \quad \leftarrow \text{LRT}$$

the threshold λ can be obtained as follows :



choose λ so that $P_F = \underbrace{\alpha'}_{\text{fixed}}$

likelihood ratio.

But

$$P_F = \int_{\lambda}^{\infty} \underbrace{P_{\Lambda|H_0}(\Lambda|H_0)\, d\Lambda}_{} = \alpha'$$

means H_0 is sent but $\Lambda > \lambda$

$\therefore$ false alarm!!

Aside:

It may not be necessary to calculate $P_{\Lambda|H_0}$ in Neyman-Pearson criterion. We may be able to find a sufficient statistic.

$$\Lambda(\underline{R}) = \frac{P_{\underline{r}|H_1}(\underline{R}|H_1)}{P_{\underline{r}|H_0}(\underline{R}|H_0)}$$

by selecting the variables properly

$$\Lambda(L,Y) = \frac{P_{l,y|H_1}(L,Y|H_1)}{P_{l,y|H_0}(L,Y|H_0)} =$$

$$= \frac{\cancel{P_{y|l,H_1}(Y|L,H_1)}\,P_l(L|H_1)}{\cancel{P_{y|l,H_0}(Y|L,H_0)}\,P_l(L|H_0)} = \frac{P_l(L|H_1)}{P_l(L|H_0)}$$

2. Consider a single measurement r=s+n, where n~(0,2) and s is a constant equal to either 0 or 1. Determine an optimum decision rule to choose between the hypotheses

H_0: s=0

H_1: s=1

Using a Neyman-Pearson test with P_F=.1.

$$H_1 : \quad r = 1 + n$$

$$H_0 : \quad r = n$$

$$p_{r|H_1}(R|H_1) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(R-1)^2}{2\sigma^2}\right)$$

$$p_{r|H_0}(R|H_0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{R^2}{2\sigma^2}\right)$$

$$\Lambda(\underline{R}) = \exp\left(-\left[\frac{(R-1)^2}{4} - \frac{R^2}{4}\right]\right) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_0$$

$$e^{\frac{R}{2}-\frac{1}{4}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_0 \implies \frac{R}{2} - \frac{1}{4} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \lambda_0$$

or $$R \underset{H_0}{\overset{H_1}{\gtrless}} 2\left(\ln \lambda_0 + \frac{1}{4}\right) = \gamma$$

$$P_F = \int_{Z_1} p_{r|H_0}(\underline{R}|H_0)\, d\underline{R} = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi(2)}} \exp\left(-\frac{R^2}{4}\right) dR = .1$$

Let $R' = \frac{R}{\sqrt{2}} \implies P_F = \underbrace{\int_{\frac{\gamma}{\sqrt{2}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{R'^2}{2}\right) dR'}_{} = .1$

$$Q\left(\frac{\gamma}{\sqrt{2}}\right) = .1 \implies \frac{\gamma}{\sqrt{2}} \simeq 1.3$$

$$\gamma \simeq 1.83$$

$$\therefore \boxed{R \underset{H_0}{\overset{H_1}{\gtrless}} 1.83}$$

Performance : Receiver operating characteristic (ROC)

To complete our discussion of binary Pbm we consider ROC.

Exa 1 (Revisited)

$$H_1 : \quad r_i = m + n_i$$

$$H_0 : \quad r_i = n_i$$

n_i, n_j, $i \neq j$ are indep

$$P_{n_i}(X) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-X^2/2\sigma^2\right)$$

$$l = \frac{1}{\sqrt{N}\,\sigma} \sum_{i=1}^{N} R_i \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{\sigma^2}{m} \ln\eta + \frac{Nm}{2}\right) \frac{1}{\sqrt{N}\,\sigma} = \frac{\sigma}{\sqrt{N}\,m} \ln\eta + \frac{\sqrt{N}\,m}{2\sigma}.$$

$$l = \frac{R_1}{\sqrt{N}\,\sigma} + \frac{R_2}{\sqrt{N}\,\sigma} + \cdots + \frac{R_N}{\sqrt{N}\,\sigma}$$

$$\sigma_l^2 = \frac{1}{N\sigma^2} E\{R_1^2\} + \cdots + \frac{1}{N\sigma^2} E\{R_N^2\}$$

(R_i's are indep)

Under H_0, R_i has zero mean and variance of σ^2

$$\therefore \sigma_l^2 = \frac{1}{N\sigma^2}\sigma^2 + \cdots\cdots + \frac{1}{N\sigma^2}\sigma^2 = 1$$

$$\therefore l_{H_0} = N(0, 1)$$

Under H_1, R_i has mean of m and variance of

$$\sigma^2 \Rightarrow \quad E\{l\} = \frac{m}{\sqrt{N}\sigma} + \cdots\cdots + \frac{m}{\sqrt{N}\sigma} = \frac{Nm}{\sqrt{N}\sigma} = \frac{\sqrt{N}m}{\sigma}$$

$$\therefore \quad \sigma_l^2 = \frac{1}{N\sigma^2}\sigma^2 + \cdots\cdots + \frac{1}{N\sigma^2}\sigma^2 = 1$$

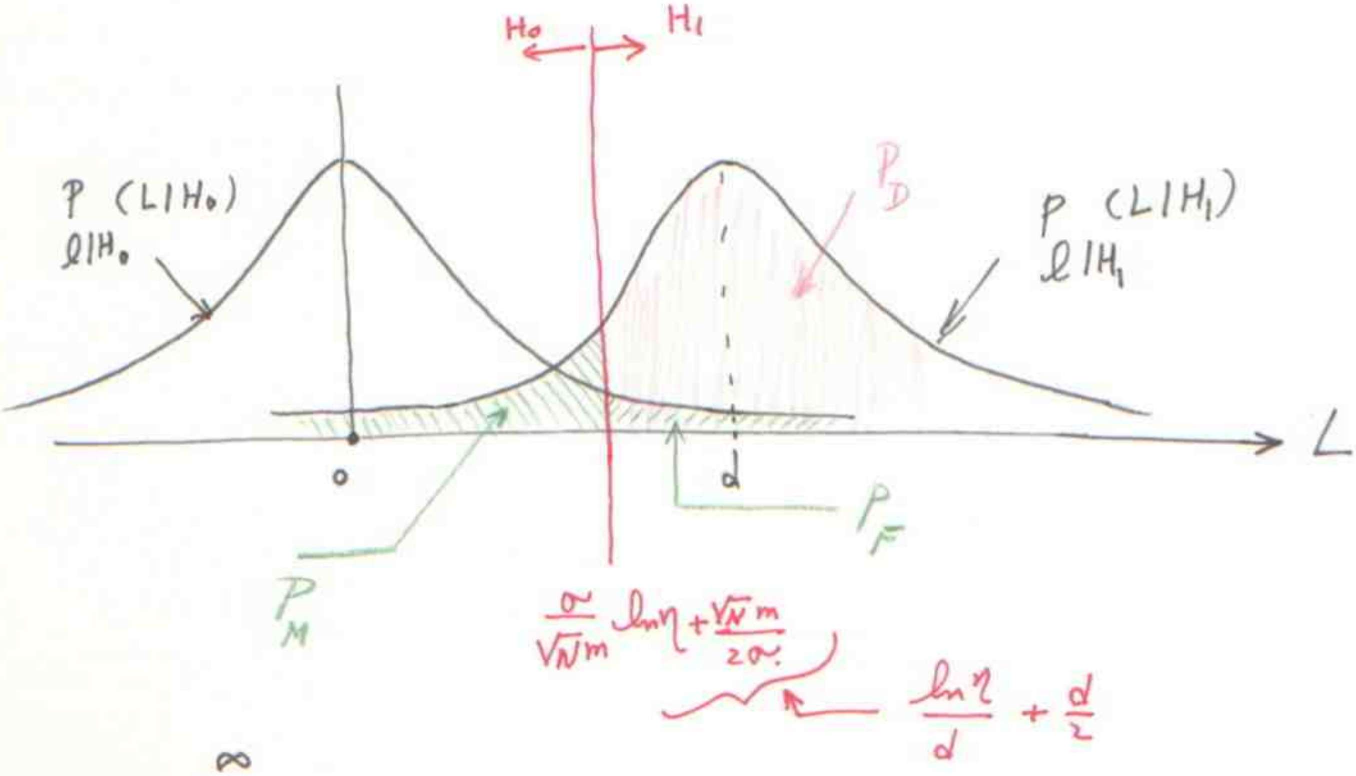
$$\therefore l_{H_1} = N\left(\frac{\sqrt{N}m}{\sigma}, 1\right).$$

$$p_{l|H_0} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{l^2}{2}\right)$$

$$p_{l|H_1} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\left[l - \overbrace{\frac{m\sqrt{N}}{\sigma}}^{d}\right]^2}{2}\right)$$

note:

$$d^2 = \frac{Nm^2}{\sigma^2} = SNR.$$



$$P_F = \int_{\frac{\ln\eta}{d}+\frac{d}{2}}^{\infty} p_{l|H_0}(L/H_0)\, dL$$

$$P_M = \int_{-\infty}^{\frac{\ln\eta}{d}+\frac{d}{2}} p_{l|H_1}(L/H_1)\, dL$$

note:

$$\operatorname{erf}_*(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2}\, dx$$

$$\operatorname{erfc}_*(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-x^2/2}\, dx.$$

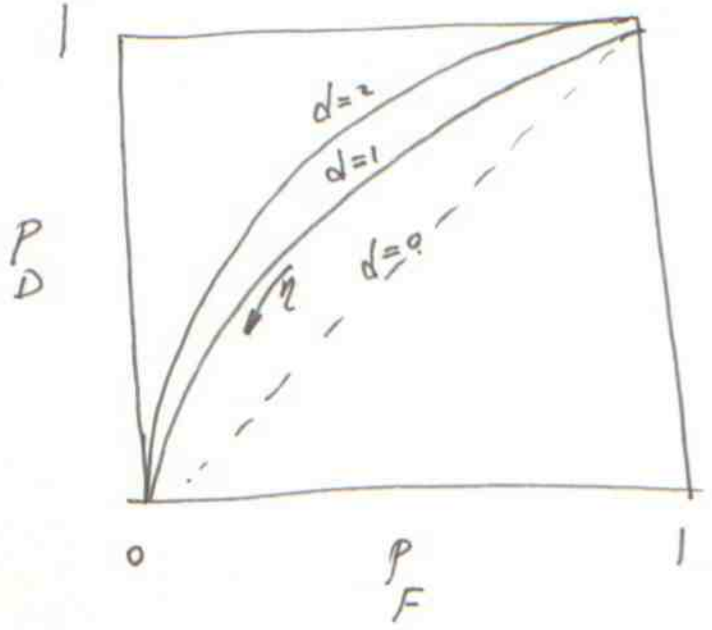
note that

$$P_F = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln\eta}{d}+\frac{d}{2}}^{\infty} e^{-y^2/2}\, dy = \operatorname{erfc}_*\left(\frac{\ln\eta}{d}+\frac{d}{2}\right).$$

$$P_D = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln\eta}{d}+\frac{d}{2}}^{\infty} \exp\left(-\frac{(x-d)^2}{2}\right) dx$$

let $y = x - d$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln\eta}{d}-\frac{d}{2}}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = \operatorname{erfc}_*\left(\frac{\ln\eta}{d}-\frac{d}{2}\right).$$



(ROC)

A special case of interest is

$$Pr(\mathcal{E}) = P_0 P_F + P_1 \overbrace{P_M}^{1-P_D}$$

the threshold for this criterion is

$$\gamma = \ln \eta = \ln P_0 - \ln(1-P_0)$$

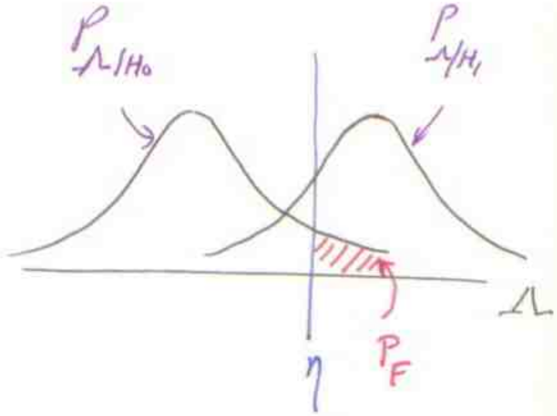
thus for specific d we can find P_D and P_F and thus $Pr(\mathcal{E})$ from ROC. ($P_D = 1 - P_M$).

Property 3 (ROC)

the slope of the ROC at a particular pnt is equal to the threshold η required to achieve P_D and P_F at that pnt.

Pf :

$$P_D = \int_{\eta}^{\infty} p_{\Lambda|H_1}(\Lambda|H_1)\, d\Lambda$$



$$P_F = \int_{\eta}^{\infty} p_{\Lambda|H_0}(\Lambda|H_0)\, d\Lambda$$

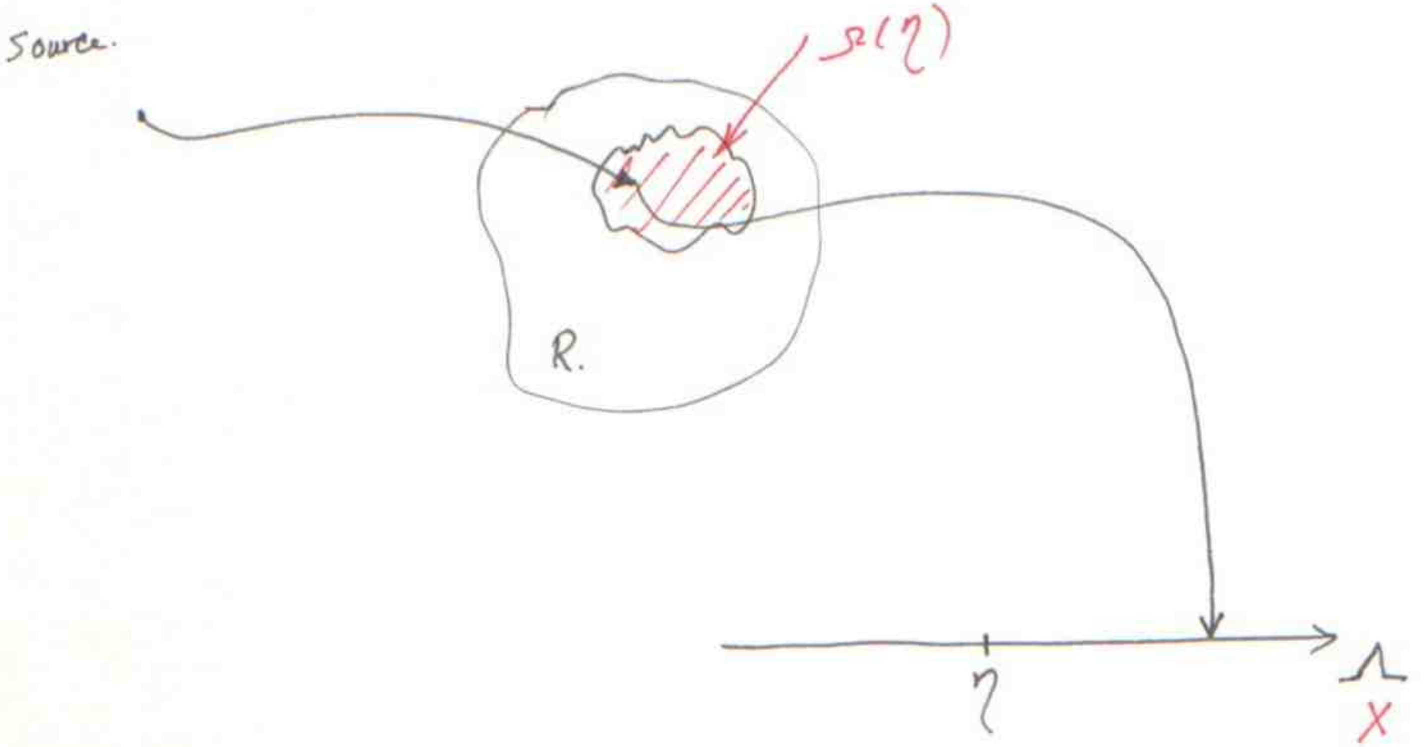
$$\frac{dP_D/d\eta}{dP_F/d\eta} = \frac{-p_{\Lambda|H_1}(\eta|H_1)}{-p_{\Lambda|H_0}(\eta|H_0)} = \frac{dP_D}{dP_F}. \qquad (1)$$

wish To show

$$\eta = \frac{p_{\Lambda|H_1}(\eta|H_1)}{p_{\Lambda|H_0}(\eta|H_0)}.$$

let

$$\Omega(\eta) \triangleq \{ \underline{R} \mid \Lambda(\underline{R}) \geq \eta \}$$



$$\therefore \quad \Omega(\eta) \triangleq \left\{ \underline{R} \;\middle|\; \frac{p_{\underline{r}|H_1}(\underline{R}|H_1)}{p_{\underline{r}|H_0}(\underline{R}|H_0)} \geq \eta \right\}$$

then

$$P_D(\eta) = P_r\{ \Lambda(\underline{R}) \geq \eta \mid H_1 \} = \int_{\Omega(\eta)} p_{\underline{r}|H_1}(\underline{R}|H_1)\, d\underline{R}$$

$$= \int_{\Omega(\eta)} \underbrace{\left[\frac{p_{\underline{r}|H_1}(\underline{R}|H_1)}{p_{\underline{r}|H_0}(\underline{R}|H_0)} \right]}_{\Lambda(\underline{R})} p_{\underline{r}|H_0}(\underline{R}|H_0)\, d\underline{R} = \int_{\Omega(\eta)} \Lambda(\underline{R})\, p_{\underline{r}|H_0}(\underline{R}|H_0)\, d\underline{R}$$

But

$$P_D(\eta) = \int_{\Omega(\eta)} \Lambda(\underline{R})\, p_{\underline{r}|H_0}(\underline{R}|H_0)\, d\underline{R} = \int_{\eta}^{\infty} X\, p_{\Lambda|H_0}(X|H_0)\, dX$$

$$\frac{dP_D(\eta)}{d\eta} = -\eta\, p_{\Lambda|H_0}(\eta|H_0)$$

$$\therefore \quad \eta = -\frac{1}{p_{\Lambda|H_0}(\eta|H_0)} \frac{dP_D}{d\eta} = -\frac{1}{p_{\Lambda|H_0}(\eta|H_0)} \left[-p_{\Lambda|H_1}(\eta|H_1) \right]. \qquad (2)$$

(1) & (2) $\Rightarrow$

$$\eta = \frac{dP_D}{dP_F}$$