A special case of interest is

\[ P_r(e) = P_o P_F + P_i P_M \]

the threshold for this criterion is

\[ \gamma = \ln \gamma = \ln P_o - \ln (1 - P_o) \]

thus for specific \( d \), we can find \( P_D \) and \( P_F \) and thus \( P_r(e) \) from ROC. (\( P_D = 1 - P_M \)).
Property 3 (ROC)

the slope of the ROC at a particular point is equal to the threshold \( \eta \) required to achieve \( P_D \) and \( P_F \) at that point.

\[
P_D = \int_{\Lambda_1 H_1}^{\infty} P_\Lambda (\Lambda | H_1) \, d\Lambda
\]

\[
P_F = \int_{\Lambda_1 H_0}^{\Lambda_2 H_0} P_\Lambda (\Lambda | H_0) \, d\Lambda
\]

\[
\frac{dP_D}{d\eta} = - \frac{P_{\Lambda_1 H_1} (\eta | H_1)}{P_{\Lambda_1 H_1}} = \frac{dP_D}{dP_F}
\]

\[
\frac{dP_F}{d\eta} = - \frac{P_{\Lambda_1 H_0} (\eta | H_0)}{P_{\Lambda_1 H_0}}
\]

wish to show

\[
\eta = \frac{P_{\Lambda_1 H_1} (\eta | H_1)}{P_{\Lambda_1 H_0} (\eta | H_0)}
\]
Let $S(\eta) = \{ R \mid \Lambda(R) \geq \eta^2 \}$.

Then

$$P_D(\eta) = P_r \{ R \mid \Lambda(R) \geq \eta^2 \} = \int_{S(\eta)} P_{R|H_1}(R|H_1) \, dR = \int_{S(\eta)} \Lambda(R) P_{R|H_0}(R|H_0) \, dR = \int_{S(\eta)} \Lambda(R) P_{R|H_0}(R|H_0) \, dR$$
But

\[
P_D(\eta) = \int \mathcal{L}(B) \frac{P_R(B|H_0)}{\Lambda H_0} \, dR = \int \frac{X \, P(x|H_0)}{\Lambda H_0} \, dx
\]

\[
\frac{dP_D(\eta)}{d\eta} = -\eta \frac{P(\eta|H_0)}{\Lambda H_0}
\]

\[
\eta = -\frac{1}{P(\eta|H_0)} \frac{dP_D}{d\eta} = -\frac{1}{P(\eta|H_0)} \left[ -P(\eta|H_0) \right]
\]

(2)

(1) \& (2) \implies \eta = \frac{dP_D}{dP_F}

Consider a case in which we must choose one of \( M \) hypotheses. In the simple \( M \)-ary test there are \( M \) source outputs, each of which corresponds to one of \( M \) hypotheses. As before, we assume that we should make a decision. Thus there are \( M^2 \) alternatives that may occur each time the experiment is conducted.

The Bayes criterion assigns a cost to each of these alternatives, assumes a set of prior probabilities \( P_0, P_1, \ldots, P_{M-1} \), and minimizes the risk.
Bayes Criterion

Let's consider \( C_{ij} \) where \( i \) signifies that the \( i \)th hypothesis has been chosen and \( j \) signifies the \( j \)th hypothesis is true. Then

\[
R = E \sum \sum c_{ij} p(D_i, H_j) = \sum \sum c_{ij} \frac{p(D_i | H_j) p(H_j)}{p(D_i | H_j) p(H_j) + p(D_i | H_0) p(H_0)} \int_{z_i} P(R | H_j) dR.
\]

To find the best test we simply vary the \( z_i \) to minimize \( R \). Let's assume that \( M = 3 \).

Then, similar to the binary case, we set

\[
R = P_0 c_{00} + P_1 c_{11} + P_2 c_{22} + \int P(C_{00} - c_{00}) P(R | H_0) dR + \int P(C_{01} - c_{01}) P(R | H_1) dR + \int P(C_{02} - c_{02}) P(R | H_2) dR.
\]

\[
+ P_1 (c_{10} - c_{11}) P(R | H_0) + P_1 (c_{12} - c_{12}) P(R | H_2) + \int P(C_{11} - c_{11}) P(R | H_1) dR + \int P(C_{10} - c_{10}) P(R | H_0) \int P(C_{12} - c_{12}) P(R | H_2) dR.
\]

\[
+ P_2 (c_{20} - c_{22}) P(R | H_0) + P_2 (c_{21} - c_{22}) P(R | H_1) + \int P(C_{21} - c_{21}) P(R | H_2) dR.
\]
clearly, we assign each \( R \) to the region in which the value of the integrand is the smallest.

Thus:

\[ R \in \mathbb{Z}_0, \text{ if } I_0(R) < I_1(R) \text{ and } I_2(R) \]

\[ R \in \mathbb{Z}, \text{ if } I_1(R) < I_0(R) \text{ and } I_2(R) \]

\[ R \in \mathbb{Z}_2, \text{ if } I_2(R) < I_0(R) \text{ and } I_1(R) \]

Define

\[ \Lambda_1(R) = \frac{P_{R \mid H_1}(R \mid H_0)}{P_{R \mid H_0}(R \mid H_0)} \quad \text{and} \quad \Lambda_2(R) = \frac{P_{R \mid H_2}(R \mid H_0)}{P_{R \mid H_0}(R \mid H_0)} \]

\[ I_0(R) \]

\[ P_{R \mid H_1}(R \mid H_2) + P_{R \mid C_0}(C_0 \mid H_1) P_{R \mid H_1}(R \mid H_1) < P_{R \mid H_0}(R \mid H_0) \]

\[ \text{(divide by } P_{R \mid H_0}(R \mid H_0)) \]

\[ I_1(R) \]
\[ P_2 (c_{02} - c_{22}) \Lambda_2 (R) + P_1 (c_{01} - c_{11}) \Lambda_1 (R) \leq P_0 (c_{10} - c_{00}) \]

\[ + P_2 (c_{12} - c_{22}) \Lambda_2 (R). \]

\[ H_0 \text{ or } H_2 \]

\[ P_1 (c_{01} - c_{11}) \Lambda_1 (R) \geq P_0 (c_{10} - c_{00}) + P_2 (c_{12} - c_{22}) \Lambda_2 (R) \]

\[ H_0 \text{ or } H_2 \]

(1)

Accordingly,

\[ H_0 \text{ or } H_1 \]

\[ P_2 (c_{02} - c_{22}) \Lambda_2 (R) \geq P_0 (c_{20} - c_{00}) + P_1 (c_{11} - c_{01}) \Lambda_1 (R) \]

\[ H_0 \text{ or } H_1 \]

(2)

\[ H_2 \text{ or } H_0 \]

\[ P_2 (c_{12} - c_{22}) \Lambda_2 (R) \geq P_0 (c_{21} - c_{01}) + P_1 (c_{11} - c_{01}) \Lambda_1 (R). \]

(3)

(Decision region)
Special Case

(similar to binary case this is equivalent to
min \( P(E) \)).

Let \( C_{00} = C_{11} = C_{22} = 0 \),
\( C_{ij} = 1 \), \( i \neq j \).

From (1), (2), and (3) we get:

\[ H_1 \text{ or } H_2 \]
\[ P_1 \Lambda_1(R) \geq P_0 \]  \hspace{1cm} (4)

\[ H_0 \text{ or } H_2 \]
\[ P_2 \Lambda_2(R) \geq P_0 \]  \hspace{1cm} (5)

\[ H_2 \text{ or } H_1 \]
\[ P_2 \Lambda_2(R) \geq P_1 \Lambda_1(R) \]  \hspace{1cm} (6)

\[ H_1 \text{ or } H_0 \]

\[ \frac{P_0}{P_1} \]

\[ \Lambda_2(R) \]

\[ \Lambda_1(R) \]

\[ H_0 \]

\[ H_1 \]

\[ H_2 \]
Let's consider (4)

\[
\frac{P_{H_1}}{P_{H_0}} \geq \frac{P_0}{P_{H_0}} \quad H_0 \text{ or } H_2
\]

\[
\therefore \quad P_{H_1} P_{H_1} (\frac{R}{H_1}) > \frac{P_0}{P_{H_0}} \quad H_0 \text{ or } H_2
\]

But

\[
P_{\frac{R}{H_1}} = \frac{P(H_1 | R) P(R)}{P(H_1)}
\]

\[
\therefore \quad P(H_1 | R) = P(R | H_1) P(H_1) / P(R)
\]

divide both sides of (7) by \( P(R) \)

\[
\begin{cases}
\frac{P(H_1 | R)}{P(H_0 | R)} \quad H_1 \text{ or } H_2 \\
\frac{P(H_0 | R)}{P(H_0 | R)} \quad H_0 \text{ or } H_2 \\
\frac{P(H_2 | R)}{P(H_0 | R)} \quad H_2 \text{ or } H_1 \\
\frac{P(H_2 | R)}{P(H_1 | R)} \quad H_2 \text{ or } H_0 
\end{cases}
\]

Similarly, from (5) and (6) we get

\[P(H_1 | R) \geq P(H_0 | R)
\]

\[P(H_0 | R) \leq P(H_0 | R)
\]
thus, an equivalent test is to compute the
a posteriori probs $P(H_0|B)$, $P(H_1|B)$, and $P(H_2|B)$
and choose the largest. This is called
a MAP detector.

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Set #2
1. 2.2.15 ( Part 1)
2. 2.2.14
3. 2.3.2
4. 2.3.3

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Set #3
1. 2.4.1.2
2. 2.4.1.3
3. 2.4.1.2
4. 2.5.1