Chapter 4

Detection of Signals
- Estimation of Signal Parameters

This chapter deals with extension of the results of the classical theory to the case in which the observations consist of conts waveforms.

We detect conts signals and estimate parameters such that "a" from \( s(t, A) \) where \( s(t) \) is known.
In this chapter we use the techniques developed in chapter 3 to reduce problems involving continuous waveforms to problems involving r.v. which we can solve by the methods of Chapter 2. The steps are:

1. Use KL expansion to map the received signal into a point in decision space (signal space). This is performed by a correlation operation and is indepent of the decision criterion used.

2. Select a decision rule and apply the techniques of Chap 2. The actual received waveform is no longer of importance.
Simple binary (AWGN)

\[ H_1 : \quad r(t) = \sqrt{E} s(t) + w(t), \quad 0 \leq t \leq T \]
\[ H_0 : \quad r(t) = w(t), \quad 0 \leq t \leq T \]

\[ \int_0^T s(t)^2 \, dt = 0 \quad \int_0^T (\sqrt{E}s(t))^2 \, dt = E \quad \text{(signal energy)} \]

Now, map \( r(t) \) into decision space and decide whether \( H_0 \) or \( H_1 \) is true. To simplify the problem,

Let

\[ r(t) = \text{l.i.m.} \sum_{i=1}^{K} r_i \phi_i(t); \quad 0 \leq t \leq T \]

\( K \to \infty \)

\[ \therefore \quad r(t) \to \bar{r} \]

Then, recognize a sufficient statistic.
Now, consider a cov set \( \{ \phi_i(t) \} \) and let
\[
\phi_i(t) = s(t).
\]
\[
H_1 : \quad r_i = \int_0^T r(t) \phi_i(t) \, dt = \int_0^T \left[ \sqrt{E} s(t) + w(t) \right] s(t) \, dt = \sqrt{E} + w_i
\]
\[
H_0 : \quad r_i = \int_0^T w(t) s(t) \, dt = w_i
\]

The remaining \( r_i \) (\( i > 1 \)) are Gaussian r.v.'s as well.

Now, we know that
\[
E \left\{ w_i w_j \right\} = 0 ; \quad i \neq j
\]
\[
\Rightarrow \{ w_i \} ; \quad i = 1, 2, \ldots \text{ are s.i.}
\]
we see that only $r_1$ affects the decision.

Note that

\[ P_{r_1 r_2 \ldots / H_0} = P_{r_1 / H_0} P_{r_2 r_3 \ldots / H_0} \]

and

\[ P_{r_1 r_2 \ldots / H_1} = P_{r_1 / H_1} P_{r_2 \ldots / H_1} \]

where

\[ P_{r_2 \ldots / H_0} = P_{r_2 \ldots / H_1} \]

\[ r_i \] is a sufficient statistic and our decision is as follows:

\[ r(t) \rightarrow S(t) \rightarrow \int_0^T \cdot \, dt \rightarrow \text{threshold} \rightarrow \text{Decision} \]

or

\[ r(t) \rightarrow h(t) \rightarrow S(T-t) \rightarrow \text{matched filter} \rightarrow \text{threshold} \rightarrow \text{Decision} \]
we wish to find the statistics of \( r_i \).

Recall

\[
H_i : r_i = \sqrt{E} + w_i
\]

\[
H_0 : r_i = w_i
\]

\[
E[r_i | H_0] = E[w_i] = 0
\]

\[
E[r_i | H_i] = \sqrt{E}
\]

\[
\text{Var} \{ r_i | H_0 \} = E[w_i^2] = E \left\{ \int_0^T \sum_{n=0}^{T} \delta(u - v) \right\}
\]

\[
= \frac{N_0}{2} \int_0^T \frac{1}{2\pi} e^{-\frac{1}{2}(u-v)^2} d\nu
= \frac{N_0}{2}
\]

\[
\text{Var} \{ r_i | H_i \} = \frac{N_0}{2}
\]

\[
\text{threshold} = \frac{\ln \xi}{\sqrt{\frac{1}{2}}}
\]
This problem is now example 1 of Chap 2, pp. 36-40
with \( m = \sqrt{E}, \quad N = 1, \quad \sigma^2 = \frac{N_0}{2} \).

\[
\therefore \quad d = \frac{\sqrt{N} m}{\sigma} = \frac{\sqrt{E}}{\sqrt{\frac{N_0}{2}}} = \sqrt{\frac{2E}{N_0}} \quad (d^2 = \text{SNR})
\]

See Figs 4.13, 4.14 P. 251, 252.

the key to the simplicity in the rule was our ability
to reduce an infinite dim. space to a one-dim.
decision space. Clearly, we should end up
with the same receive if we do not recognize
that a sufficient statistic is available!!

To do this we construct \( \Lambda(B) \) directly.
The curves in Figs. 2.9a and 2.9b of Chapter 2 are directly applicable and are reproduced as Figs. 4.13 and 4.14. We see that the performance depends only on the received signal energy $E$ and the noise spectral height $N_0$—the signal shape is not important. This is intuitively logical because the noise is the same along any coordinate.

The key to the simplicity in the solution was our ability to reduce an infinite dimensional observation space to a one-dimensional decision space by exploiting the idea of a sufficient statistic. Clearly, we should end up with the same receiver even if we do not recognize that a sufficient statistic is available. To demonstrate this we construct the likelihood ratio directly. Three observations lead us easily to the solution.

1. If we approximate $r(t)$ in terms of some finite set of numbers,
Fig. 4.14 Probability of detection vs \( \left( \frac{2E}{N_0} \right)^{\frac{1}{2}} \).

\( r_1, \ldots, r_K, (r) \), we have a problem in classical detection theory that we can solve.

2. If we choose the set \( r_1, r_2, \ldots, r_K \) so that

\[
p_{r_1, r_2, \ldots, r_K \mid H_i}(R_1, R_2, \ldots, R_K \mid H_i) = \prod_{j=1}^{K} p_{r_j \mid H_i}(R_j \mid H_i), \quad i = 0, 1, \tag{12}
\]

that is, the observations are conditionally independent, we have an easy problem to solve.
\[ r(t) = \sum_{i=1}^{K} r_i \phi_i(t) \]

\[ r_i = \int_{0}^{T} r(t) \phi_i(t) \, dt \]

\[ i = 1, 2, \ldots, K \]

Know:

\[ H_1 : \quad r(t) = \sqrt{E}s(t) + w(t) \]

\[ H_0 : \quad r(t) = w(t) \]

\[ H_1 : \quad r_i = \int_{0}^{T} \sqrt{E}s(t) \phi_i(t) \, dt + \int_{0}^{T} w(t) \phi_i(t) \, dt = s_i + w_i \]

\[ H_0 : \quad r_i = \int_{0}^{T} w(t) \phi_i(t) \, dt = w_i \]

The coefficients \( s_i \) correspond to an expansion of the signal

\[ s_k(t) = \sum_{i=1}^{K} s_i \phi_i(t) \]

and

\[ \sqrt{E}s(t) = \lim_{K \to \infty} s_k(t) \]

\[ E\{s(t) + w(t)\} = s(t) + w(t) \]

\[ E\{s_i\} = 0 \quad E\{w_i\} = 0 \]

\( r_i \) are uncorrelated
\[ E[r_i | H_0] = 0, \quad \text{Var} \{ r_i | H_0 \} = \frac{N_0}{2} \]
\[ E[r_i | H_1] = s_i, \quad \text{Var} \{ r_i | H_1 \} = \frac{N_0}{2} \]

The \( r_i \)'s are uncorrelated because \( W(t) \) is white. They are also independent because \( W(t) \) is Gaussian.

\[ \Lambda_k(r_k(t)) = \Lambda_k(r_k) = \frac{P_{H_1}(R | H_1)}{P_{H_0}(R | H_0)} = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi N_0}} \exp\left( -\frac{(R_i - s_i)^2}{2N_0} \right) \]

This is \( r_k \), for simplicity, \( k \) is dropped!

\[ \ln \Lambda_k(r_k) = \frac{2}{N_0} \sum_{i=1}^{K} R_i s_i - \frac{1}{N_0} \sum_{i=1}^{K} s_i^2 \]

But,
\[ \int_0^T r_k(t) s_k(t) \, dt = \int_0^T \left[ \sum_{i=1}^{K} R_i \phi_i(t) \right] \left[ \sum_{j=1}^{K} s_j \phi_j(t) \right] \, dt = \sum_{i=1}^{K} R_i s_i, \]
and
\[ \int_0^T s_k^2(t) \, dt = \int_0^T \left[ \sum_{i=1}^{K} s_i \phi_i(t) \right] \left[ \sum_{j=1}^{K} s_j \phi_j(t) \right] \, dt = \sum_{i=1}^{K} s_i^2. \]
\[
\ln \Lambda (r_k) = \frac{2}{N_0} \int_0^T r_k(t) s_k(t) \, dt - \frac{1}{N_0} \int_0^T s_k^2(t) \, dt
\]

\[
limit_{k \to \infty} \ln \Lambda (r_k) = \ln \Lambda [r(t)] = \frac{2}{N_0} \int_0^T \sqrt{E} r(t) s(t) \, dt - \frac{1}{N_0} \int_0^T s(t)^2 \, dt
\]

\[
\begin{align*}
H_1 & \geq \ln \eta \\
H_0 & \leq \ln \eta
\end{align*}
\]

\[
\frac{2\sqrt{E}}{N_0} \int_0^T r(t) s(t) \, dt \geq \frac{E}{N_0} + \ln \eta
\]

Once again, we set a correlation receiver!

Remark: Note that the PDF \( P \left( R | H_0 \right) \) or \( P \left( R | H_1 \right) \) are not well defined, but the likelihood ratio is \( ! \). For example,

\[
P \left( R | H_0 \right) = \prod_{i=1}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp \left( - \frac{1}{2} \frac{R_i^2}{N_0/2} \right) = \frac{1}{(2\pi N_0)^{N/2}} \exp \left\{ - \frac{1}{2} \sum_{i=1}^{\infty} R_i^2 \right\}
\]

But \( \sum_{i=1}^{\infty} R_i^2 = \sum_{i=1}^{\infty} w_i^2 \to \infty \) (white noise)