General Binary Problem in WGN

\[ H_1 : \quad r(t) = \sqrt{E_1} s_1(t) + w(t) \quad \text{\(0 \leq t \leq T\)} \]

\[ H_0 : \quad r(t) = \sqrt{E_0} s_0(t) + w(t) \]

\[ \int_0^T s_1(t)^2 dt + \int_0^T s_0(t)^2 dt = 1 \]

\[ \int_0^T s_1(t) s_0(t) dt = \rho \quad \text{\(|\rho| \leq 1\)} \]

Choose

\[ \Phi_1(t) = s_1(t) \]

\[ \Phi_2(t) = \frac{1}{\sqrt{1-\rho^2}} \left[ s_0(t) - \rho s_1(t) \right] \]

The remaining \(\Phi_i\)'s are chosen arbitrarily to give a cons set, but we don't need to find them.

This is obtained by subtracting out the comp of \(s_0(t)\) that is correlated with \(s_1(t)\) and normalizing the results.
\[ V(t) = \sum_{i=1}^{\infty} r_i \phi_i(t) \]

\[ r_i = \int_0^T r(t) \phi_i(t) \, dt \]

\[ H_i : r_i = \int_0^T r(t) \phi_i(t) \, dt \]

\[ = \int_0^T \left( \sqrt{E} s_i(t) + w(t) \right) \phi_i(t) \, dt \]

\[ = \begin{cases} 
\int_0^T \sqrt{E} s_i(t) \phi_i(t) \, dt + w_i, & i = 1 \\
\int_0^T \sqrt{E} s_i(t) \phi_i(t) \, dt + w_i, & i = 2 \\
\int_0^T \sqrt{E} s_i(t) \phi_i(t) \, dt + w_i, & i \geq 3 
\end{cases} \]

\[ = \begin{cases} 
\sqrt{E} s_i(t) + w_i, & i = 1 \\
w_i, & i = 2 \\
w_i, & i \geq 3 
\end{cases} \]
\[ r_i = \int_0^T r(t) \phi_i(t) \, dt \]

\[ = \int_0^T \left[ \sqrt{E_0} s_0(t) + w(t) \right] \phi_i(t) \, dt \]

\[ = \begin{cases} 
\int_0^T \sqrt{E_0} s_0(t) \phi_i(t) \, dt + w_1 & ; i = 1 \\
\int_0^T \sqrt{E_0} s_0(t) \phi_i(t) \, dt + w_2 & ; i = 2 \\
\int_0^T \sqrt{E_0} s_0(t) \phi_i(t) \, dt + w_i & ; i \geq 3 
\end{cases} \]

\[ = \begin{cases} 
\delta_{01} + w_1 & ; i = 1 \\
\delta_{02} + w_2 & ; i = 2 \\
w_k & ; i \geq 3 
\end{cases} \]
$r_i \quad i \geq 2$ does not depend on which hypothesis.

Thus:

$r_1$ and $r_i$ are sufficient statistic.

$r = (r_1, r_i)$

$S_0 = E\{ r | H_0 \}$

$S_1 = E\{ r | H_1 \}$
\[ \text{Var} \{ r_i | H_0 \} = \text{Var} \{ r_i | H_1 \} = \frac{N_0}{2} \]

Now find the covariance of \( r_1 \) and \( r_2 \)

\[ H_j: \quad E \left\{ (r_i - \mu_i)(r_j - \mu_j) | H_j \right\} = E \left\{ \int_0^T \phi(u) \phi(u) \, du \right\} \]

\[ = \frac{N_0}{2} \int_0^T \phi_1(t) \phi_2(t) \, dt = 0 \]

\[ \frac{P_r | H_1}{P_r | H_0} = \Lambda(B) = \frac{1}{\sqrt{2 \pi N_0}} \frac{1}{\sqrt{2 \pi N_0}} \frac{1}{\sqrt{2 \pi N_0}} \frac{1}{\sqrt{2 \pi N_0}} \exp \left[ -\frac{(R_1 - s_{11})^2}{2N_0} \right] \exp \left[ -\frac{(R_1 - s_{12})^2}{2N_0} \right] \exp \left[ -\frac{(R_2 - s_{01})^2}{2N_0} \right] \exp \left[ -\frac{(R_2 - s_{02})^2}{2N_0} \right] \]

\[ \ln \Lambda(B) = -\frac{1}{N_0} (R_1 - s_{11})^2 - \frac{1}{N_0} (R_1 - s_{12})^2 + \frac{1}{N_0} (R_1 - s_{01})^2 + \frac{1}{N_0} (R_2 - s_{02})^2 \]

\[ R = \left( \begin{array}{c} R_1 \\ R_2 \end{array} \right), \quad S_1 = \left[ \begin{array}{c} s_{11} \\ s_{12} \end{array} \right], \quad S_0 = \left( s_{01} \right) \]
\[
\ln \Lambda(R) = -\frac{1}{N_0} [R - s_1]^T [R - s_1] + \frac{1}{N_0} [R - s_0]^T [R - s_0]
\]

\[
= -\frac{1}{N_0} \left[ R^T R - R^T s_1 - s_1^T R + s_1^T s_1 \right] + \frac{1}{N_0} \left[ R^T R - R^T s_0 - s_0^T R + s_0^T s_0 \right]
\]

\[
= \frac{2}{N_0} \left[ R^T (s_1 - s_0) \right] - \frac{1}{N_0} \left[ \|s_1\|^2 - 1 \right] \|s_0\|^2 \]

\[
\geq \ln \eta
\]

\[
\Rightarrow \quad \frac{R^T (s_1 - s_0)}{H_0} \geq \frac{1}{2} \left[ \|s_1\|^2 - 1 \right] + \frac{N_0}{2} \ln \eta = k
\]

\textbf{inner pivot of the received signal vector with the difference in the components!}

\textbf{fixed!}

\textbf{Decision line}

\textbf{Say H_0}

\textbf{Say H_1}

\textbf{the new coordinates using transformation! only l is useful for decision!}
Aside

Know \( P^T (s_1 - s_0) = k \). Wish to find the value \( m \). First, we show that \( D \perp (s_1 - s_0) \).

Pf. \( \tilde{R}^T_1 (s_1 - s_0) = k \) \( = D \) \( (\tilde{R} - \tilde{R}_0)^T (s_1 - s_0) = 0 \)
\( \tilde{R}^T_2 (s_1 - s_0) = k \)
\( \therefore \tilde{R}_1 - \tilde{R}_2 \perp s_1 - s_0 \)

Now, we find \( m \). Recall that

\[
\ln A(R) = -\frac{1}{N_0} (R_1 - s_{11})^2 - \frac{1}{N_0} (R_2 - s_{12})^2 + \frac{1}{N_0} (R_1 - s_{01})^2 + \frac{1}{N_0} (R_2 - s_{02})^2 \geq \ln \eta
\]

Hence,
\[
\therefore -\frac{1}{N_0} |R - s_{11}|^2 + \frac{1}{N_0} |R - s_{01}|^2 = \ln \eta
\]
\[
|R - s_{50}|^2 - |R - s_{51}|^2 = N_0 \ln \eta
\]
\[ d_1^2 - d_2^2 = N_0 \ln \eta \]

But,

\[ d_1^2 = \eta^2 + \mu^2 \]

\[ d_2^2 = \eta^2 + (\eta - \mu)^2 = \eta^2 + \mu^2 + \mu^2 - 2\mu \eta \]

Hence,

\[ d_1^2 - d_2^2 = 2\mu \eta - \mu^2 = N_0 \ln \eta \]

\[ \eta = \frac{N_0 d_1^2 + d^2}{2d} \]

Note that for \( \eta = 1 \rightarrow D \rightarrow M = \frac{d}{2} \)

this is a typical case for binary digital communication systems and leads to a "bisection" decision rule. The receiver under these circumstances can be interpreted as a "minimum-distance receiver" and

\[ P_r(c) = P \left( \ln \left( \frac{C_1}{C_0} \right) \right) = e^{-\ln \left( \frac{C_1}{C_0} \right) + \frac{d}{2}} \]

\[ \eta = 1 \]
Remarks about the noise

1. Noise components along r₁ and r₂ (φ₁, φ₂) are independent and also have the same distribution.

2. Noise components along y and l will still be independent because $P_{1H_0}$ (IR $1H_0$) has symmetry about $s₀$ and $P_{1H_1}$ (IR $1H_1$) has symmetry about $s₁$.

The decision can be made by generating $l$, the new sufficient statistic in $(s₁ - s₀)$ direction. That is, we need to design our receiver based on the component of $R$ in $(s₁ - s₀)$ direction. This component can be obtained by taking the inner product of $r(t)$ with the normalized difference between $\sqrt{E₁} s₁(t)$ and $\sqrt{E₀} s₀(t)$.

$$S_{Δ}(t) = \sqrt{E₁} s₁(t) - \sqrt{E₀} s₀(t)$$

$$Q_{Δ}(t) = \frac{S_{Δ}(t)}{(E₁ - 2P\sqrt{E₁E₀} + E₀)^{1/2}}$$
\[ E\{ L|H_i \}^2 = E\left\{ \int_0^T r(t) p_{\Delta}(t) \, dt \right\} \]

\[ E\{ L|H_i \} = \frac{E_1 - \sqrt{E_0 E_1} \rho}{(E_1 - 2\rho \sqrt{E_0 E_1} + E_0)^{1/2}} \]

\[ E\{ L|H_0 \} = \frac{\sqrt{E_0 E_1} \rho - E_0}{(E_1 - 2\rho \sqrt{E_0 E_1} + E_0)^{1/2}} \]

\[ \text{Var}\{ L|H_i \} = \frac{N_0}{\sigma^2}, \quad i = 1, 2 \]

\[ d^2 = \frac{2}{N_0} \left( E_1 + E_0 - 2\rho \sqrt{E_0 E_1} \right) \]

\[ P_F = \text{erfc} \left( \frac{d}{\sqrt{2}} \right) \]

\[ P_D = \text{erfc} \left( \frac{d}{\sqrt{2}} - \frac{d}{\sigma} \right) \]
\[ E\{L|H_1\} - E\{L|H_0\} = (E_1 - 2 \sqrt{E_0 E_1 (P + E_0)} \frac{1}{2} \]

Normalize \( l \) to unit variance, then \( d \) becomes the separation of the means in the normalized \( l \).

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**Notes**

1. The best signal choice appears to be when \( P = -1 \), the performance is indy of signal shape and depends only on energies.

2. If our prior is to minimize \( P_E \) and \( P_0 = P \), then the decision line in \( x_1; x_2 \) space is the perpendicular bisector of \( S_1 - S_2 \). If \( E_0 = E_1 \), the line passes through the origin.
In this case we can choose the signal which is closest to \( r(t) \) or the one that has more (clarity). 

Correlation