2. Finite observation interval; stationary noise process.
Problem Solution 4.2.4

1. Denote the output of the feedback system under $H_0$ as $y_0(t)$ and under $H_1$ as $y_1(t)$. Since $x(t)$ is deterministic, we find $y_i(t)$ using transform techniques. The closed loop transfer function on the $i$-th hypothesis is,

$$\frac{Y_i(s)}{X(s)} = \frac{A_i}{s + A_i}, \quad i = 0, 1.$$ 

Thus,

$$Y_i(s) = \left(\frac{A_i}{s + A_i}\right) X(s), \quad i = 0, 1,$$

or in the time domain

$$y_1(t) = x(t) \otimes A_1 u_{-1}(t) e^{-A_1 t} = \int_0^t A_1 x(\tau) e^{-A_1(t-\tau)} d\tau, \quad t > 0, \quad i = 0, 1.$$ 

To find the likelihood ratio test, we use the results on pp. 254–256 directly. Writing (4.31c) in terms of waveforms

$$ t[y_1(t) - y_0(t)] dt \gg \frac{N_0}{2} \ln n + \frac{1}{2} \left[ \int_0^T y_1^2(t) dt - \int_0^T y_0^2(t) dt \right], $$

which is the required result. The performance is completely determined by $d^2$. From (2.36)

$$d^2 = \frac{2}{N_0} (E_1 + E_0 - 2\rho \sqrt{E_1 E_0}),$$

where
\[ E_1 = \int_0^T y_1^2(t) \, dt \]
\[ E_0 = \int_0^T y_0^2(t) \, dt \]
\[ \sqrt{E_1 E_0} = \int_0^T y_0(t) y_1(t) \, dt. \]

2. When \( x(t) \) is an impulse,
\[ y_1(t) = A_1 e^{-A_1 t}, \quad t > 0, \]
\[ y_0(t) = A_0 e^{-A_0 t}, \quad t > 0. \]

Then,
\[ E_1 = \int_0^\infty A_1^2 e^{-2A_1 t} \, dt = \frac{A_1}{2} \]

and
\[ \sqrt{E_1 E_0} = \int_0^\infty A_1 A_0 e^{-2(A_1+A_0) t} \, dt = \frac{A_0 A_1}{2(A_1+A_0)} \]

\[ d^2 = \frac{2}{N_0} \left[ -\frac{A_1 + A_0}{2} - \frac{A_0 A_1}{A_1 + A_0} \right]. \]

To find \( P_D \) and \( P_F \), we use this value of \( d^2 \) in (2.37) and (2.38).
Problem Solution 4.2.25

This problem is similar to Example 1 on p. 276. As in (112),

\[ \ln \Lambda_1[r(t), A, B] + \ln p_a(A) + \ln p_b(B) \]

\[ = \frac{2}{N_0} \int_{-T}^{T} B s(t-A) r(t) \, dt - \frac{B^2}{N_0} \int_{-T}^{T} s^2(t-A) \, dt \]

\[ - \frac{B^2}{2\sigma_b^2} + k. \quad \text{(P.1)} \]

To do this we observe that

\[ \max_{A,B} [\ln \Lambda_1(r(t), A, B)] = \max_A [\max_B \ln \Lambda_1(r(t), A, B)] \]

\[ = \max_A [\ln \Lambda_1(r(t), A, b_{\text{map}}(A))]. \]

Thus, we can find \( b_{\text{map}}(A) \) and then maximize over \( A \).

First, we find a necessary condition on \( b_{\text{map}} \) by differentiating and equating the result to zero.

\[ \left\{ \frac{2}{N_0} \int_{-T}^{T} B s(t-A) r(t) \, dt - 2B \left( \frac{1}{N_0} \int_{-T}^{T} s^2(t-A) \, dt \right) \right\}_{B=b_{\text{map}}} = 0. \]

This can be solved for \( b_{\text{map}} \),

\[ b_{\text{map}} = \frac{2}{N_0} \int_{-T}^{T} s(t-A) r(t) \, dt - \frac{2E_s}{N_0} + \frac{1}{2\sigma_b^2} \quad \text{(P.2)} \]
where $E$ is the signal energy. Since the solution is unique and the second derivative is negative we have the absolute maximum for any value of $A$. To find $\hat{b}_{\text{map}}$ we substitute (P.2) in (P.1),

$$\ln \Lambda_1[r(t), A, \hat{b}_{\text{map}}] + \ln p_A(A) + \ln p_B(\hat{b}_{\text{map}})$$

$$= \left( \int_{-T}^{T} s(t-A) r(t) \, dt \right)^2 \left( \frac{2k_o}{N_0} - \frac{E_s k_o^2}{N_0} - \frac{1}{2\sigma_b^2} \right) + k.$$ 

Note that $\ln p_A(A)$ is constant. $\ln p_B(\hat{b}_{\text{map}})$ is contained in the gain multiplying the first term.

To find $\hat{b}_{\text{map}}$, we must generate this as a function of $A$ and choose the maximum. The optimum receiver is shown in the figure on page 109. When the output assumes its maximum value, we observe the output of the $k_o$ gain to obtain $\hat{b}_{\text{map}}$.
SAMPLE AT $t = \hat{a}_{map}$. This value is $b_{map}$.
1. From (4.153) we have,

$$L_n A(r(t)) \Delta L(T_f) = \int_0^{T_f} S(\tau) Q_n(\tau, u) r(u) d\tau du.$$ 

We have let $E = 1$ and absorbed the constant term in the bias. From Problem 4.3.3,

$$L(T_f) = \frac{2}{N_0} \int_0^{T_f} S(\tau) \delta(\tau - u) - h_o(\tau, u; T_f) r(u) d\tau du,$$

or

$$L(T_f) = \frac{2}{N_0} \int_0^{T_f} S(\tau) \tau d\tau - \frac{2}{N_0} \int_0^{T_f} S(\tau) h_o(\tau, u; T_f) r(u) d\tau du. \quad (P.1)$$

2. Now, write

$$L(T_f) = \int_{T_i}^{T_f} \frac{\partial L(t)}{\partial t} dt, \quad (P.2)$$

where $L(t)$ is obtained by letting $T_f = t$ in (P.1).

$$L(t) = \frac{2}{N_0} \int_{T_i}^{t} S(\tau) \tau d\tau - \frac{2}{N_0} \int_{T_i}^{t} S(\tau) h_o(\tau, u; t) r(u) d\tau du. \quad (P.3)$$

Differentiating (P.3) with respect to $t$ gives,

$$\frac{\partial L(t)}{\partial t} = \frac{2}{N_0} [S(t) r(t) - \int_{T_i}^{t} S(t) h_o(t, u; t) r(u) du]$$

$$- \int_{T_i}^{t} \tau d\tau [S(\tau) h_o(\tau, t; t) r(t) + \int_{T_i}^{t} S(\tau) \frac{h_o(\tau, u; t)}{\partial t} r(u) du]. \quad (P.4)$$
Now substituting

$$\frac{\partial h_o(\tau,u:t)}{\partial t} = -h_o(\tau,t:t)h_o(t,u:t), \quad (P.5)$$

using $h_o(\tau,t:t) = h_o(t,\tau:t)$ which follows from the symmetric property of the inverse kernel, we obtain:

$$\frac{\partial L(t)}{\partial t} = \frac{2}{N_0} \left[ s(t)r(t)-s(t) \int_0^t h_o(t,u:t)r(u)du \right. \right.$$

$$\left. - r(t) \int_0^t h_o(t,u:t)s(u)du \right.$$

$$\left. + \int_0^t h_o(t,u:t)r(u)du \int_0^t h_o(t,\tau:t)s(\tau)d\tau \right]$$

$$= \frac{2}{N_0} \left[ s(t) - \int_0^t h_o(t,u:t)s(u)du \right] \cdot$$

$$\left[ r(t) - \int_0^t h_o(t,u:t)r(u)du \right]. \quad (P.6)$$

Using (P.6) in (P.2)

$$L(T_f) = \frac{2}{N_0} \int_{T_1}^{T_f} \left[ s(t) - \int_0^t h_o(t,u:t)s(u)du \right] \cdot$$

$$\left[ r(t)- \int_0^t h_o(t,u:t)r(u)du \right] dt. \quad (P.7)$$

3. (P.7) can be rewritten as,
L(T_f) = \int_{T_1}^{T_f} dt \left[ \frac{2}{N_0} \int_{T_1}^{T_f} du [\delta(t-u) - h_0(t,u:t)] s(u) \right]
\cdot \left[ \frac{2}{N_0} \int_{T_1}^{T_f} d\tau [\delta(t-\tau) - h_0(t,\tau:t)] r(\tau) \right]. \quad (P.8)

This can be written as,

L(T_f) = \int_{T_1}^{T_f} dt \int_{T_1}^{T_f} h_{wr}(t,u) s(u) du \int_{T_1}^{T_f} h_{wr}(t,\tau) r(\tau) d\tau, \quad (P.9)

where we have defined

h_{wr}(t,\tau) = \frac{2}{N_0} [\delta(t-\tau) - h_0(t,\tau:t)]. \quad (P.10)

Note that (P.9) is the correct result (there was an error in the problem statement). We refer to the filter described by (P.10) as a realizable whitening filter. It is easy to verify that its output is white when the input is n(t).

4. Clearly, the integral equation for h_0(t,\tau:t) is the same as that for h_0(t,\tau) when the interval endpoint T_f = t. Thus,

\frac{N_0}{2} h_0(t,\tau:t) + \int_{T_1}^{t} h_0(t,u:t) K_c(u,\tau) du = K_c(t,\tau), \quad T_1 \leq \tau \leq t.
Problem Solution 4.3.12

The easiest way to solve this problem is to use a whitening approach (see pp. 290-297). The noise spectrum is,

\[ S_n(\omega) = \frac{2\alpha}{\omega^2 + \alpha^2}. \]

This is identical to the noise in (241). The whitening filter is,

\[ H_w(j\omega) = \frac{1}{\sqrt{2\alpha}} (j\omega + \alpha). \]

Note that the whitening filter operates only on the present so the restriction of a finite observation interval is unimportant (see comment 2 on p. 314).

The output of \( H_w(j\omega) \) when the input is \( \sin^2 \left( \frac{2\pi t}{T} \right) \) is

\[
s_1^*(t) = \frac{1}{\sqrt{2\alpha}} \left[ \frac{d}{dt} \left( \sin^2 \left( \frac{2\pi t}{T} \right) + \alpha \sin^2 \left( \frac{2\pi t}{T} \right) \right) \right]
\]

\[
= \frac{1}{\sqrt{2\alpha}} \left[ -\frac{4\pi}{T} \sin \left( \frac{2\pi t}{T} \right) \cos \left( \frac{2\pi t}{T} \right) + \alpha \sin^2 \left( \frac{2\pi t}{T} \right) \right].
\]

(P.1)

Also,

\[ s_{21}^*(t) = -s_{12}^*(t). \]

Now, from (215),

\[ d^2 = \int_0^T s_{\Delta^*}^2(t) \, dt, \]

or

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\[ d^2 = 4 \int_0^T s_1^2(t) \, dt. \quad (P.2) \]

Substituting (P.1) in (P.2) and integrating gives,

\[ d^2 = \frac{2}{\alpha} \left[ \frac{3T}{8} \alpha^2 + \frac{T}{2} \left( \frac{2\pi}{T} \right)^2 \right]. \]

This expression is minimized if

\[ \alpha = \frac{4\pi}{\sqrt{3}T}. \quad (P.3) \]

Since \( \Pr(\varepsilon) \) is related monotonically to \( d^2 \), the value of \( \alpha \) in (P.3) maximizes the opposition's error probability.