Chapter 10

Sinusoidal steady-state power

Calculations
Average Power and RMS Values

Nearly all problems in circuit analysis are concerned with applying one or more sources of electric energy to a circuit and then quantitatively determining one or more responses throughout the circuit. A response may be a current or a voltage, but we are also interested in the amount of energy supplied from the sources, in the amount of energy dissipated or stored within the circuit, and in the manner in which energy is delivered to the points at which the responses are determined. Primarily, however, we are concerned with the rate at which energy is being generated and absorbed; our attention must now be directed to power.

We shall begin by considering instantaneous power, the product of the time-domain voltage and time-domain current associated with the element or network of interest. The instantaneous power is sometimes quite useful in its own right, because its maximum value might have to be limited in order to avoid exceeding the safe or useful operating range of a physical device. For example, transistor and vacuum-tube power amplifiers both produce a distorted output, and speakers give a distorted sound, when the peak power exceeds a certain limiting value. However, we are mainly interested in instantaneous power for the simple reason that it provides us with the means to calculate a more important quantity, the average power. In a similar way, the progress of cross-country automobile trip is best described by the average velocity; our interest in the instantaneous velocity is limited to the avoidance of maximum velocities which will endanger our safety or arouse the highway patrol.

In practical problems we shall deal with values of average power which range from the small fraction of a picowatt available in a telemetry signal from outer space, to the few watts of audio power supplied to the speakers in a high-fidelity stereo system, the several hundred watts required to invigorate the morning coffeepot, or the 10 billion watts generated at the Grand Coulee Dam.

Our discussion will not be concerned entirely with the average power delivered by a sinusoidal current or voltage; we shall therefore define a quantity called the effective value, a mathematical measure of the effectiveness of other waveforms in delivering power. Our study of power will be completed by considering the descriptive quantities power factor and complex power, two concepts which introduce the practical and economic aspects associated with the distribution of electric power.
In order to determine the power absorbed by the inductor, we first obtain the inductor voltage:

\[ v_L = L \frac{di}{dt} \]

(1)

\[ = V_0 e^{-Rt/L}i(t) + \frac{LV_0}{R} (1 - e^{-Rt/L}) \frac{du(t)}{dt} \]

\[ = V_0 e^{-Rt/L}i(t) \]

(2)

since \( du(t)/dt \) is zero for \( t > 0 \) and \( (1 - e^{-Rt/L}) \) is zero at \( t = 0 \). The power absorbed by the inductor is thus

\[ p_L = v_i i = \frac{V_0^2}{R} e^{-Rt/L} (1 - e^{-Rt/L}) u(t) \]

(3)

Only a few algebraic manipulations are required to show that

\[ p = p_R + p_L \]

which serves to check the accuracy of our work.

The majority of the problems which involve power calculations are perhaps those which deal with circuits excited by sinusoidal forcing functions in the steady state. As we have been told previously, even when periodic nonsinusoidal forcing functions are employed, it is possible to resolve the problem into a number of subproblems in which the forcing functions are sinusoidal. The special case of the sinusoid therefore deserves special attention.

Let us change the voltage source in the circuit of Fig. 10-1 to the sinusoidal source \( V_m \cos \omega t \). The familiar time-domain response is

\[ i(t) = I_m \cos (\omega t + \phi) \]

where

\[ I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \quad \text{and} \quad \phi = -\tan^{-1} \frac{\omega L}{R} \]

The instantaneous power delivered to the entire circuit in the sinusoidal steady state is, therefore,

\[ p = vi = V_m I_m \cos (\omega t + \phi) \cos \omega t \]

which we shall find convenient to rewrite in a form obtained by using the trigonometric identity for the product of two cosine functions. Thus,

\[ p = \frac{V_m I_m}{2} [\cos (2\omega t + \phi) + \cos \phi] \]

\[ = \frac{V_m I_m}{2} \cos \phi + \frac{V_m I_m}{2} \cos (2\omega t + \phi) \]

The last equation possesses several characteristics which are true in general for circuits in the sinusoidal steady state. One term, the first, is not a function of time; and a second term is included which has a cyclic variation at twice the applied frequency. Since this term is a cosine wave, and since sine waves and cosine waves have average values which are zero (when averaged over an integral number of periods), this introductory example may serve to indicate
and then by integrating from some other time \( t \), to \( t_1 + T \):

\[
P_s = \frac{1}{T} \int_{t_1}^{t_1+T} p(t) \, dt
\]

The equality of \( P_1 \) and \( P_s \) should be evident from the graphical interpretation of the integrals; the area which represents the integral to be evaluated in determining \( P_s \) is smaller by the area from \( t_1 \) to \( t_s \), but greater by the area from \( t_1 + T \) to \( t_s + T \). The periodic nature of the curve requires these two areas to be equal. Thus, the average power may be computed by integrating the instantaneous power over any interval which is one period in length and then dividing by the period:

\[
P = \frac{1}{T} \int_{t_1}^{t_1+T} p \, dt \tag{7}
\]

It is important to note that we might also integrate over any integral number of periods, provided that we divide by this same integral number of periods. Thus,

\[
P = \frac{1}{nT} \int_{t_1}^{t_1+nT} p \, dt \quad n = 1, 2, 3, \ldots \tag{8}
\]

If we carry this concept to the extreme by integrating over all time, another useful result is obtained. We first provide ourselves with symmetrical limits on the integral

\[
P = \frac{1}{nT} \int_{-nT/2}^{nT/2} p \, dt
\]

and then take the limit as \( n \) becomes infinite,

\[
P = \lim_{n \to \infty} \frac{1}{nT} \int_{-nT/2}^{nT/2} p \, dt
\]

If \( p(t) \) is a mathematically well-behaved function, as all physical forcing functions and responses are, it is apparent that if a large integer \( n \) is replaced by a slightly larger number which is not an integer, then the value of the integral \( P \) and of \( p \) is changed by a negligible amount; moreover, the error decreases as \( n \) increases. Without justifying this step rigorously, we therefore replace the discrete variable \( nT \) by the continuous variable \( \tau \):

\[
P = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} p \, dt \tag{9}
\]

We shall find it convenient on several occasions to integrate periodic functions over this "infinite period." Examples of the use of Eqs. (7), (8), and (9) follow.

Let us illustrate the calculation of the average power of a periodic wave by finding the average power delivered to a resistor \( R \) by the (periodic) sawtooth current waveform shown in Fig. 10-3a. We have

\[
i(t) = \frac{I_n}{T} t \quad 0 < t \leq T
\]

\[
i(t) = \frac{I_n}{T} (t - T) \quad T < t \leq 2T
\]
Figure 10-3

(a) A sawtooth current waveform and (b) the instantaneous power waveform it produces in a resistor $R$.

\[ p(t) = \frac{1}{T^2} I_m^2 R t^2 \quad 0 < t \leq T \]

\[ p(t) = \frac{1}{T^2} I_m^2 R (t - T)^2 \quad T < t \leq 2T \]

and so on, as sketched in Fig. 10-3b. Integrating over the simplest range of one period, from $t = 0$ to $t = T$, we have

\[ P = \frac{1}{T} \int_0^T \frac{1}{T^2} I_m^2 R t^2 \, dt = \frac{1}{3} I_m^2 R \]

The selection of other ranges of one period, such as from $t = 0.1T$ to $t = 1.1T$, would produce the same answer. Integration from 0 to $2T$ and division by $2T$—that is, the application of Eq. (8) with $n = 2$ and $t_0 = 0$—would also provide the same answer.

Now let us obtain the general result for the sinusoidal steady state. We shall assume the general sinusoidal voltage

\[ v(t) = V_m \cos(\omega t + \theta) \]

and current

\[ i(t) = I_m \cos(\omega t + \phi) \]

associated with the device in question. The instantaneous power is

\[ p(t) = V_m I_m \cos(\omega t + \theta) \cos(\omega t + \phi) \]

Again expressing the product of two cosine functions as one-half the sum of the cosine of the difference angle and the cosine of the sum angle,

\[ p(t) = \frac{1}{2} V_m I_m \cos(\theta - \phi) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta + \phi) \quad (10) \]

we may save ourselves some integration by an inspection of the result. The first term is a constant, independent of $t$. The remaining term is a cosine function; $p(t)$ is therefore periodic, and its period is $1T$. Note that the period $T$ is associated with the given current and voltage, and not with the power; the power function has a period $1T$. However, we may integrate over an interval of $T$ to determine the average value if we wish; it is necessary only that we also divide by $T$. Our familiarity with cosine and sine waves, however, shows that the average value of either over a period is zero. There is thus no need to integrate Eq. (10) formally; by inspection, the average value of the second term is zero over a
period \( T \) (or \( \frac{1}{2} T \)) and the average value of the first term, a constant, must be that constant itself. Thus,

\[
P = \frac{1}{2} V_a I_a \cos(\theta - \phi)
\]

This important result, introduced in the previous section for a specific circuit, is therefore quite general for the sinusoidal steady state. The average power is one-half the product of the crest amplitude of the voltage, the crest amplitude of the current, and the cosine of the phase-angle difference between the current and the voltage; the sense of the difference is immaterial.

Let us try our hands on a numerical example.

**Example 10-1** We are given the time-domain voltage \( v = 4 \cos(\pi t/6) \) V, and we wish to find the power relationships that result when the corresponding phasor voltage \( V = 4/0^\circ \) V is applied across an impedance \( Z = 2/60^\circ \) \( \Omega \).

**Solution:** The phasor current is \( V/Z = 2/-60^\circ \) A, and the average power is

\[
P = \frac{1}{2}(4)(2) \cos 60^\circ = 2 \text{ W}
\]

The time-domain voltage,

\[
v(t) = 4 \cos \left( \frac{\pi t}{6} \right)
\]

time-domain current,

\[
i(t) = 2 \cos \left( \frac{\pi t}{6} - 60^\circ \right)
\]

and instantaneous power,

\[
p(t) = 8 \cos \left( \frac{\pi t}{6} \right) \cos \left( \frac{\pi t}{6} - 60^\circ \right) = 2 + 4 \cos \left( \frac{\pi t}{3} - 60^\circ \right)
\]

are all sketched on the same time axis in Fig. 10-4. Both the 2-W average value of the power and its period of 6 s, one-half the period of either the current or the voltage, are evident. The zero value of the instantaneous power at each instant when either the voltage or current is zero is also apparent.

![Figure 10-4](image_url)

Curves of \( v(t) \), \( i(t) \), and \( p(t) \) are plotted as functions of time for a simple circuit in which the phasor voltage \( V = 4/0^\circ \) V is applied to the impedance \( Z = 2/60^\circ \) \( \Omega \) at \( \omega = \pi/6 \text{ rad/s} \).
Two special cases are worth isolating for consideration, the average power delivered to an ideal resistor, and that to an ideal reactor (any combination of only capacitors and inductors).

The phase-angle difference between the current through and the voltage across a pure resistor is zero, and therefore

\[ P_R = \frac{1}{2} V_m I_m \]

or

\[ P_R = \frac{1}{2} I_m^2 R \] (12)

or

\[ P_R = \frac{V_m^2}{2R} \] (13)

The last two formulas, enabling us to determine the average power delivered to a pure resistance from a knowledge of either the sinusoidal current or voltage, are simple and important. They are often misused. The most common error is made in trying to apply them in cases where, say, the voltage included in Eq. (13) is not the voltage across the resistor. If care is taken to use the current through the resistor in Eq. (12) and the voltage across the resistor in Eq. (13), satisfactory operation is guaranteed. Also, do not forget the factor of \( \frac{1}{2} \).

The average power delivered to any device which is purely reactive must be zero. This is evident from the 90° phase difference which must exist between current and voltage; hence, \( \cos(\theta - \phi) = 0 \) and

\[ P_X = 0 \]

The average power delivered to any network composed entirely of ideal inductors and capacitors is zero; the instantaneous power is zero only at specific instants. Thus, power flows into the network for a part of the cycle and out of the network during another portion of the cycle, with no power lost.

**Example 10-2** Find the average power being delivered to an impedance \( Z_L = 8 - j11 \) \( \Omega \) by a current \( I = 5/20^\circ \) A.

**Solution:** We may find the solution quite rapidly by using Eq. (12). Only the 8-Ω resistor enters the average-power calculation, and

\[ P = \frac{1}{2}(5)^2 8 = 100 \text{ W} \]

since no average power can be absorbed by the \(-j11 \) \( \Omega \). Note also that if the current is given in rectangular form, say, \( I = 2 + j5 \) A, then the magnitude squared is \( 2^2 + 5^2 \), and the average power delivered to \( Z_L = 8 - j11 \) \( \Omega \) would be

\[ P = \frac{1}{2}(2^2 + 5^2)8 = 116 \text{ W} \]

**Example 10-3** As a further example illustrating these power relationships, let us consider the circuit shown in Fig. 10-5. We need the average power absorbed by each of the three passive elements and the average power supplied by each source.

**Solution:** The values of \( I_1 \) and \( I_2 \) are found by any of several methods, such as mesh analysis, nodal analysis, or superposition. They are

\[ I_1 = 5 - j10 = 11.18^\circ / -63.4^\circ \]

\[ I_2 = 5 - j5 = 7.07^\circ / -45^\circ \]
The downward current through the 2-Ω resistor is
\[ I_1 - I_2 = -j5 = 5/90^\circ \]
so that \( I_m = 5 \) A, and the average power absorbed by the resistor is found most easily by Eq. (12):
\[ P_R = |I_m|^2 R = (5^2)2 = 25 \text{ W} \]
This result may be checked by using Eq. (11) or Eq. (13). The average power absorbed by each reactive element is zero.

We next turn to the left source. The voltage 20/0° and current 11.18/63.4° satisfy the active sign convention, and thus the power delivered by this source is
\[ P_{\text{left}} = \overline{20}(11.18) \cos (0° + 63.4°) = 50 \text{ W} \]
In a similar manner, we find the power absorbed by the right source,
\[ P_{\text{right}} = \overline{10}(7.07) \cos (0° + 45°) = 25 \text{ W} \]
Since 50 = 25 + 25, the power relations check.

In Sec. 2-5 of Chap. 2, we considered the maximum power transfer theorem as applied to resistive loads and resistive source impedances. For a Thévenin source \( V \), and impedance \( Z_{\text{th}} = R_{\text{th}} + jX_{\text{th}} \) connected to a load \( Z_L = R_L + jX_L \), it may be shown readily (see Prob. 10 in this chapter) that the average power delivered to the load is a maximum when \( R_L = R_{\text{th}} \) and \( X_L = -X_{\text{th}} \), that is, when \( Z_L = Z_{\text{th}}^* \). This result is often dignified by calling it the maximum power transfer theorem for the sinusoidal steady state:

An independent voltage source in series with an impedance \( Z_{\text{th}} \) or an independent current source in parallel with an impedance \( Z_{\text{th}} \) delivers a maximum average power to that load impedance \( Z_L \) which is the conjugate of \( Z_{\text{th}} \), or \( Z_L = Z_{\text{th}}^* \).

It is apparent that the resistive condition considered in Chap. 2 is merely a special case.

We must now pay some attention to nonperiodic functions. One practical example of a nonperiodic power function for which an average power value is desired is the power output of a radio telescope which is directed toward a radio star. Another is the sum of a number of periodic functions, each function having a different period, such that no greater common period can be found for the combination. For example, the current
\[ i(t) = \sin t + \sin nt \] (14)
is nonperiodic because the ratio of the periods of the two sine waves is an irrational number. At \( t = 0 \), both terms are zero and increasing. But the first term is zero and increasing only when \( t = 2\pi n \), where \( n \) is an integer, and thus periodicity demands that \( nt \) or \( \pi(2\pi n) \) must equal \( 2\pi m \), where \( m \) is also an integer. No solution (integral values for both \( m \) and \( n \)) for this equation is possible. It may be illuminating to compare the nonperiodic expression in Eq. (14) with the periodic function

\[
i(t) = \sin t + \sin 3.14t
\]

where 3.14 is an exact decimal expression and is not to be interpreted as \( 3.141592 \ldots \). With a little effort, it can be shown that the period of this current wave is \( 100\pi \) s.\(^1\)

The average value of the power delivered to a 1-\( \Omega \) resistor by either a periodic current such as Eq. (15) or a nonperiodic current such as Eq. (14) may be found by integrating over an infinite interval. Much of the actual integration can be avoided because of our thorough knowledge of the average values of simple functions. We therefore obtain the instantaneous power delivered by the current in Eq. (14) by applying Eq. (9):

\[
P = \lim_{R \to \infty} \frac{1}{R} \int_{-R}^{R} (\sin^2 t + \sin^2 \pi t + 2 \sin t \sin \pi t) \, dt
\]

We now consider \( P \) as the sum of three average values. The average value of \( \sin^2 t \) over an infinite interval is found by replacing \( \sin^2 t \) by \( (1 - \frac{1}{2} \cos 2t) \); the average is obviously \( \frac{1}{2} \). Similarly, the average value of \( \sin^2 \pi t \) is also \( \frac{1}{2} \). And the last term can be expressed as the sum of two cosine functions, each of which must certainly have an average value of zero. Thus,

\[
P = \frac{1}{2} + \frac{1}{2} + 0 = 1 \text{ W}
\]

An identical result is obtained for the periodic current, Eq. (15).

Applying this same method to a current function which is the sum of several sinusoids of different periods and arbitrary amplitudes,

\[
i(t) = I_{n1}\cos \omega_1 t + I_{n2}\cos \omega_2 t + \cdots + I_{nN}\cos \omega_N t
\]

we find the average power delivered to a resistance \( R \),

\[
P = k(I_{n1}^2 + I_{n2}^2 + \cdots + I_{nN}^2)R
\]

The result is unchanged if an arbitrary phase angle is assigned to each component of the current. This important result is surprisingly simple when we think of the steps required for its derivation: squaring the current function, integrating, and taking the limit. The result is also just plain surprising because it shows that, in this special case of a current such as Eq. (16), superposition is applicable to power. Superposition is not applicable for a current which is the sum of two direct currents, nor is it applicable for a current which is the sum of two sinusoids of the same frequency.

**Example 10-4** Find the average power delivered to a 4-\( \Omega \) resistor by the current

\[
i_1 = 2 \cos 10t - 3 \cos 20t \text{ A.}
\]

\(^1\) \( T_1 = 2\pi \) and \( T_2 = 2\pi/3.14 \). Therefore we seek integral values of \( m \) and \( n \) such that \( 2\pi m = 2\pi = 2\pi n/3.14 \) or \( 3.14n = m \), or \( 100n = m \), or \( 157n = 50m \). Thus, the smallest integral values for \( m \) and \( n \) are \( n = 50 \) and \( m = 157 \). The period is therefore \( T = 2\pi n = 100\pi \), or \( T = 2\pi (157/3.14) \approx 100\pi \) s.
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Solution: Since the two cosine terms are at different frequencies, the two average-power values may be calculated separately and added. Thus, this current delivers \( \frac{1}{2} (2^2)4 + \frac{1}{2} (3^2)4 = 8 + 18 = 26 \text{ W} \) to a 4-Ω resistor.

Example 10-5 Find the average power delivered to a 4-Ω resistor by the current
\[ i = 2 \cos 10t - 3 \cos 10t \text{ A}. \]

Solution: Here, the two components of the current are at the same frequency, and they must therefore be combined into a single sinusoid at that frequency. Thus, \( i_2 = 2 \cos 10t - 3 \cos 10t = -\cos 10t \) delivers only \( \frac{1}{2} (1^2)4 = 2 \text{ W} \) of average power to that same resistor.

10-3. Determine the average power delivered to each of the boxed networks in the circuit of Fig. 10-6.

\[ \text{Ans: 208 W; 61.5 W; 123.1 W} \]

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10-4. Find the average power delivered to a 5-Ω resistor by each of the periodic waveforms shown in Fig. 10-7.

\[ \text{Ans: 208 W; 333 W; 160.0 W} \]

Figure 10-6

See Drill Prob. 3.

Figure 10-7

See Drill Probs. 4 and 6, and Prob. 16.
10-4 Effective values of current and voltage

Most of us are aware that the voltage available at the power outlets in our homes is a sinusoidal voltage having a frequency of 60 Hz and a “voltage” of 115 V. But what is meant by “115 volts”? This is certainly not the instantaneous value of the voltage, for the voltage is not a constant. The value of 115 V is also not the amplitude which we have been symbolizing as \( V_a \); if we displayed the voltage waveform on a calibrated cathode-ray oscilloscope, we should find that the amplitude of this voltage at one of our ac outlets is \( 115 \sqrt{2} \), or 162.6 V. We also cannot fit the concept of an average value to the 115 V, because the average value of the sine wave is zero. We might come a little closer by trying the magnitude of the average over a positive or negative half cycle; by using a rectifier-type voltmeter at the outlet, we should measure 103.5 V. As it turns out, however, the 115 V is the effective value of this sinusoidal voltage. This value is a measure of the effectiveness of a voltage source in delivering power to a resistive load.

Let us now proceed to define the effective value of any periodic waveform representing either a current or a voltage. We shall consider the sinusoidal waveform as only a special, albeit practically important, case. Let us arbitrarily define effective value in terms of a current waveform, although a voltage could equally well be selected.

The effective value of any periodic current is equal to the value of the direct current which, flowing through an \( R \)-ohm resistor, delivers the same average power to the resistor as does the periodic current.

In other words, we allow the given periodic current to flow through the resistor, determine the instantaneous power \( i^2 R \), and then find the average value of \( i^2 R \) over a period; this is the average power. We then cause a direct current to flow through this same resistor and adjust the value of the direct current until the same value of average power is obtained. The resulting magnitude of the direct current is equal to the effective value of the given periodic current. These ideas are illustrated in Fig. 10-8.

The general mathematical expression for the effective value of \( i(t) \) is not...
The average power delivered to the resistor by the periodic current $i(t)$ is

$$ P = \frac{1}{T} \int_0^T i^2 R \, dt = R \frac{\int_0^T i^2 \, dt}{T} $$

where the period of $i(t)$ is $T$. The power delivered by the direct current is

$$ P = I_{\text{dc}}^2 R $$

Equating the power expressions and solving for $I_{\text{dc}}$, we get

$$ I_{\text{dc}} = \sqrt{\frac{1}{T} \int_0^T i^2 \, dt} \quad (18) $$

The result is independent of the resistance $R$, as it must be to provide us with a worthwhile concept. A similar expression is obtained for the effective value that we can assign to a periodic voltage by replacing $i$ and $I_{\text{dc}}$ by $v$ and $V_{\text{dc}}$, respectively.

We notice that the effective value is obtained by first squaring the time function, then taking the average value of the squared function over a period, and finally taking the square root of the average function. In other words, we take the (square) root of the mean of the square; for this reason, the effective value is often called the root-mean-square value, or simply the rms value.

The most important special case is that of the sinusoidal waveform. Let us select the sinusoidal current

$$ i(t) = I_m \cos (\omega t + \phi) $$

with a period $T = \frac{2\pi}{\omega}$. We can substitute in Eq. (18) to obtain the effective value

$$ I_{\text{dc}} = \sqrt{\frac{1}{T} \int_0^T I_m^2 \cos^2(\omega t + \phi) \, dt} $$

$$ = I_m \sqrt{\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left[ \frac{1}{2} + \frac{1}{2} \cos(2\omega t + 2\phi) \right] dt} $$

$$ = I_m \sqrt{\frac{\omega}{4\pi} \left[ \frac{t}{2} \right]_{0}^{2\pi/\omega}} = \frac{I_m}{\sqrt{2}} \quad // $$
Thus the effective value of a sinusoidal current is a real quantity which is independent of the phase angle and numerically equal to $1/\sqrt{2} = 0.707$ times the amplitude of the current. A current $\sqrt{2} \cos(\omega t + \phi)$ A, therefore, has an effective value of 1 A and will deliver the same power to any resistor as will a direct current of 1 A.

It should be noted carefully that the $\sqrt{2}$ factor that we obtained as the ratio of the amplitude of the periodic current to the effective value is applicable only when the periodic function is sinusoidal. For the sawtooth waveform of Fig. 10-3, for example, the effective value is equal to the maximum value divided by $\sqrt{3}$. The factor by which the maximum value must be divided to obtain the effective value depends on the mathematical form of the given periodic function; it may be either rational or irrational, depending on the nature of the function.

The use of the effective value also simplifies slightly the expression for the average power delivered by a sinusoidal current or voltage by avoiding use of the factor $\frac{1}{2}$. For example, the average power delivered to an $R$-ohm resistor by a sinusoidal current is

$$P = \frac{1}{2} I_{\text{rms}}^2 R$$

Since $I_{\text{rms}} = I_{\text{m}}/\sqrt{2}$, the average power may be written as

$$P = I_{\text{rms}}^2 R$$

(19)

The other power expressions may also be written in terms of effective values:

$$P = V_{\text{rms}} I_{\text{rms}} \cos(\theta - \phi)$$

(20)

$$P = \frac{V_{\text{rms}}^2}{R}$$

(21)

The fact that the effective value is defined in terms of an equivalent dc quantity provides us with average power formulas for resistive circuits which are identical with those used in dc analysis.

Although we have succeeded in eliminating the factor $\frac{1}{2}$ from our average power relationships, we must now take care to determine whether a sinusoidal quantity is expressed in terms of its amplitude or its effective value. In practice, the effective value is usually used in the fields of power transmission and distribution and of rotating machinery; in the areas of electronics and communications, the amplitude is more often used. We shall assume that the amplitude is specified unless the term rms is explicitly used.

In the sinusoidal steady state, phasor voltages and currents may be given either as effective values or as amplitudes; the two expressions differ only by a factor $\sqrt{2}$. The voltage 50/30° V is expressed in terms of an amplitude; as an rms voltage, we should describe the same voltage as 35.4/30° V rms.

In order to determine the effective value of a periodic or nonperiodic waveform which is composed of the sum of a number of sinusoids of different frequencies, we may use the appropriate average-power relationship of Eq. (17), developed in the previous section, rewritten in terms of the effective values of the several components:

$$P = (I_{\text{rms}}^1 + I_{\text{rms}}^2 + \cdots + I_{\text{rms}}^k) R$$

(22)

These results indicate that if a sinusoidal current of 5 A rms at 60 Hz flows through a 2-Ω resistor, an average power of $5^2(2) = 50$ W is absorbed by the resistor; if a second current—say, 3 A rms at 120 Hz—is also present, the
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absorbed power is \(3^2(2) + 50 = 68\) W; however, if the second current is also at 60 Hz, then the absorbed power may have any value between 8 W and 128 W, depending on the relative phase of the two current components.

We therefore have found the effective value of a current which is composed of any number of sinusoidal currents of different frequencies,

\[
I_{\text{eff}} = \sqrt{I_1^2_{\text{eff}} + I_2^2_{\text{eff}} + \cdots + I_n^2_{\text{eff}}}
\]  

(23)

The total current may or may not be periodic; the result is the same. The effective value of the sum of the 60- and 120-Hz currents in the preceding example is 5.83 A; the effective value of the sum of the two 60-Hz currents may have any value between 2 and 8 A.

Drill Problems

0-6. Use Eq. (18) to calculate the effective values of the three periodic waveforms shown in Fig. 10-7.

\[\text{Ans: } 6.45\ \text{A}; 8.16\ \text{A}; 5.66\ \text{A}\]

0-7. Calculate the effective value of each of the periodic voltages: (a) 6 \(\cos 25t\); (b) 6 \(\cos 25t + 4 \sin (25t + 30^\circ)\); (c) 6 \(\cos 25t + 5 \cos^2 25t\); (d) 6 \(\cos 25t + \sin 30t + 4\) V.

\[\text{Ans: } 4.24\ \text{V}; 6.16\ \text{V}; 5.23\ \text{V}; 6.82\ \text{V}\]

Historically, the introduction of the concepts of apparent power and power factor can be traced to the electric power industry, where large amounts of electric energy must be transferred from one point to another; the efficiency with which this transfer is effected is related directly to the cost of the electric energy, which is eventually paid by the consumer. Customers who provide loads which result in a relatively poor transmission efficiency must pay a greater price for each kilowatt-hour (kWh) of electric energy they actually receive and use. In a similar way, customers who require a costlier investment in transmission and distribution equipment by the power company will also pay more for each kilowatt-hour unless the company is benevolent and enjoys losing money.

Let us first define apparent power and power factor and then show briefly how these terms are related to the aforementioned economic situations. We shall assume that the sinusoidal voltage

\[v = V_n \cos (\omega t + \theta)\]

is applied to a network and the resultant sinusoidal current is

\[i = I_n \cos (\omega t + \phi)\]

The phase angle by which the voltage leads the current is therefore \((\theta - \phi)\). The average power delivered to the network, assuming a passive sign convention at its input terminals, may be expressed either in terms of the maximum values:

\[P = IV_n I_n \cos (\theta - \phi)\]

or in terms of the effective values:

\[P = \left(\frac{V_{\text{eff}}}{\sqrt{2}}\right) I_{\text{eff}} \cos (\theta - \phi)\]

If our applied voltage and current responses had been dc quantities, the average power delivered to the network would have been given simply by the product of the voltage and the current. Applying this dc technique to the sinusoidal
problem, we should obtain a value for the absorbed power which is "apparently"
given by the familiar product $V_{av}I_{av}$. However, this product of the effective
values of the voltage and current is not the average power; we define it as the
apparent power. Dimensionally, apparent power must be measured in the same
units as real power, since $\cos (\theta - \phi)$ is dimensionless; but in order to avoid
confusion, the term voltamperes, or VA, is applied to the apparent power. Since
$\cos (\theta - \phi)$ cannot have a magnitude greater than unity, it is evident that the
magnitude of the real power can never be greater than the magnitude of the
apparent power.

Apparent power is not a concept which is limited to sinusoidal forcing
functions and responses. It may be determined for any current and voltage
waveshapes by simply taking the product of the effective values of the current
and voltage.

The ratio of the real or average power to the apparent power is called the
power factor, symbolized by PF. Hence,

$$\text{PF} = \frac{\text{average power}}{\text{apparent power}} = \frac{P}{V_{av}I_{av}}$$

In the sinusoidal case, the power factor is simply $\cos (\theta - \phi)$, where $(\theta - \phi)$ is
the angle by which the voltage leads the current. This relationship is the reason
why the angle $(\theta - \phi)$ is often referred to as the PF angle.

For a purely resistive load, the voltage and current are in phase, $(\theta - \phi)$ is
zero, and the PF is unity. The apparent power and the average power are
equal. Unity PF, however, may also be achieved for loads which contain both
inductance and capacitance if the element values and the operating frequency
are selected to provide an input impedance having a zero phase angle.

A purely reactive load, that is, one containing no resistance, will cause a
phase difference between the voltage and current of either plus or minus 90°,
and the PF is therefore zero.

Between these two extreme cases there are the general networks for which
the PF can range from zero to unity. A PF of 0.5, for example, indicates a load
having an input impedance with a phase angle of either 60° or −60°; the former
describes an inductive load, since the voltage leads the current by 60°, while
the latter refers to a capacitive load. The ambiguity in the exact nature of the
load is resolved by referring to a leading PF, or a lagging PF, the terms leading
or lagging referring to the phase of the current with respect to the voltage. Thus,
an inductive load will have a lagging PF and a capacitive load a leading PF.

Before considering the practical consequences of these ideas, we can prac-
tice the necessary techniques by analyzing the circuit illustrated in Fig. 10-9.

**Example 10-9** Calculate values for the average power delivered to each of the
two loads shown in Fig. 10-9, the apparent power supplied by the source, and
the power factor of the combined loads.

---

**Figure 10-9**

A circuit in which we seek the average power delivered to each element, the apparent power supplied by the source, and the power factor of the combined load.

![Diagram](image-url)
Solution: The source current is

\[ I_s = \frac{60/0^\circ}{2 - j1 + 1 + j5} = 12/ -53.1^\circ \text{ A (rms)} \]

Therefore, the source supplies an average power of

\[ P_s = (60)(12) \cos[0^\circ - (-53.1^\circ)] = 432 \text{ W} \]

The upper load receives an average power

\[ P_{upper} = 12^2(2) = 288 \text{ W} \]

For the right-hand load, we find an average power of

\[ P_{right} = 12^2(1) = 144 \text{ W} \]

Thus, the source provides 432 W, of which 288 W is dissipated in the upper load and 144 W in the load to the right. The power balance is correct.

The source supplies an apparent power of 60(12) = 720 VA.

Finally, the power factor of the combined loads is found by considering the voltage and current associated with the combined loads. This power factor is, of course, identical to the power factor for the source. Thus,

\[ PF = \frac{P}{V \cdot I_{eff}} = \frac{432}{60(12)} = 0.6 \]

We might also combine the two loads in series, obtaining \((3 + j4) \Omega\) or \(5/53.1^\circ \Omega\). We therefore identify 53.1° as the PF angle, and thus have a PF of \(\cos 53.1^\circ = 0.6\). We also note that the combined load is inductive, and the PF is therefore 0.6 lagging.

The practical importance of these new terms is shown by several different situations we shall see in the following. Let us first assume that we have a sinusoidal ac generator, which is a rotating machine driven by some other device whose output is a mechanical torque, such as a steam turbine, an electric motor, or an internal-combustion engine. We shall let our generator produce an output voltage of 200 V rms at 60 Hz. Suppose now that an additional rating of the generator is stated as a maximum power output of 1 kW. The generator would therefore be capable of delivering an rms current of 5 A to a resistive load. If, however, a load requiring 1 kW at a lagging power factor of 0.5 is connected to the generator, then an rms current of 10 A is necessary. As the PF decreases, greater and greater currents must be delivered to the load if operation at 200 V and 1 kW is to be maintained. If our generator were correctly and economically designed to furnish safely a maximum current of 5 A, then these greater currents would cause unsatisfactory operation, such as causing the insulation to overheat and begin smoking, which could be injurious to its health.

The rating of the generator is more informatively given in terms of apparent power in voltamperes. Thus a 1000-VA rating at 200 V indicates that the generator can deliver a maximum current of 5 A at rated voltage; the power it delivers depends on the load, and in an extreme case might be zero. An apparent power rating is equivalent to a current rating when operation is at a constant voltage.

When electric power is being supplied to large industrial consumers by a power company, the company will frequently include a PF clause in its rate
schedules. Under this clause, an additional charge is made to the consumer whenever the PF drops below a certain specified value, usually about 0.85 lagging. Very little industrial power is consumed at leading PFs, because of the nature of the typical industrial loads. There are several reasons that force the power company to make this additional charge for low PFs. In the first place, it is evident that larger current-carrying capacity must be built into its generators in order to provide the larger currents that go with lower-PF operation at constant power and constant voltage. Another reason is found in the increased losses in its transmission and distribution system.

As an example, let us suppose that a certain consumer is using an average power of 11 kW at unity PF and 220 V rms. We also assume a total resistance of 0.2 Ω in the transmission lines through which the power is delivered to the consumer. An rms current of 50 A therefore flows in the load and in the lines, producing a line loss of 500 W. In order to supply 11 kW to the consumer, the power company must generate 11.5 kW (at the higher voltage, 230 V). Since the energy is necessarily metered at the location of each consumer, this consumer would be billed for 95.6 percent of the energy which the power company actually produced.

Now let us hypothesize another consumer, also requiring 11 kW, but at a PF angle of 60° lagging. This customer forces the power company to push 100 A through the load and (of particular interest to the company) through the line resistance. The line losses are now found to be 2 kW, and the customer’s meter indicates only 84.6 percent of the actual energy generated. This figure depart from 100 percent by more than the power company will tolerate; this costs it money. Of course, the transmission losses might be reduced by using heavier transmission lines, which have lower resistance, but this costs more money too. The power company’s solution to this problem is to encourage operation at PFs which exceed 0.9 lagging, by offering slightly reduced rates, and to discourage operation at PFs which are less than 0.85 lagging, by invoking increased rates.

The power drawn by most homes is used at reasonably high PFs (and reasonably small power levels); no charge is usually made for low-PF operation.

Besides paying for the actual energy consumed and for operation at excessively low PFs, industrial consumers are also billed for inordinate demand.3 An energy of 100 kWh is delivered much more economically at 5 kW for 20 h than if it is as 20 kW for 5 h, particularly if everyone else demands large amounts of power at the same time.

**Drill Problem**

10-8. A 440-V rms source supplies power to a load $Z_L = 10 + j2 \Omega$ through an transmission line having a total resistance of 1.5 Ω. Find (a) the average apparent power supplied to the load; (b) the average and apparent power lost in the transmission line; (c) the average and apparent power supplied by the source; (d) the power factor at which the source operates. (e) Does the average power supplied by the source equal the sum of the average powers lost in the transmission line and delivered to the load? (f) Does the apparent power supplied by the source equal the sum of the apparent powers lost in the transmission line and delivered to the load?

**Ans:** 14 209 W, 14 491 VA; 2131 W, 2131 VA; 16 341 W, 16 586 VA; 0.985 lag; yes; no.

---

3 Several electrical utilities have made experimental installations of demand kilowatthour meters on residential customer premises.
Some simplification in power calculations is achieved if power is considered to be a complex quantity. The magnitude of the complex power will be found to be the apparent power, and the real part of the complex power will be shown to be the (real) average power. The new quantity, the imaginary part of the complex power, we shall call reactive power.

We define complex power with reference to a general sinusoidal voltage $V_{\text{eff}}$ at an angle of $\theta$ across a pair of terminals and a general sinusoidal current $I_{\text{eff}} = I_{\text{eff}} \angle \phi$ flowing into one of the terminals in such a way as to satisfy the passive sign convention. The average power $P$ absorbed by the two-terminal network is thus

$$P = V_{\text{eff}} I_{\text{eff}} \cos (\theta - \phi)$$

Complex nomenclature is next introduced by making use of Euler's formula in the same way as we did in introducing phasors. We express $P$ as

$$P = V_{\text{eff}} I_{\text{eff}} \Re \{ e^{j(\theta - \phi)} \} = \Re \{ V_{\text{eff}} I_{\text{eff}} e^{-j\phi} \}$$

The phasor voltage may now be recognized as the first two factors within the brackets in the preceding equation, but the second two factors do not quite match 100% of the voltage. We must therefore make use of conjugate notation:

$$I^*_{\text{eff}} = I_{\text{eff}} e^{-j\phi}$$

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$$I^*_{\text{eff}} = I_{\text{eff}} e^{-j\phi}$$

and we may now let power become complex by defining the complex power $S$ as an excess over the real power $P$.

$$S = V_{\text{eff}} I_{\text{eff}}$$

If we first inspect the polar or exponential form of the complex power,

$$S = V_{\text{eff}} I_{\text{eff}} e^{j(\theta - \phi)}$$

we see that the magnitude of $S$ is the apparent power and the angle of $S$ is the PF angle, that is, the angle by which the voltage leads the current.

In rectangular form, we have

$$S = P + jQ$$

where $P$ is the real average power, as before. The imaginary part of the complex power is symbolized as $Q$ and is termed the reactive power. The dimensions of $Q$ are obviously the same as those of the real power $P$, the complex power $S$, and the apparent power $|S|$. In order to avoid confusion with these other quantities, the unit of $Q$ is defined as the var (abbreviation VAR), standing for volt-amperes reactive. From Eq. (24), it is seen that

$$Q = V_{\text{eff}} I_{\text{eff}} \sin (\theta - \phi)$$
Another interpretation of the reactive power may be seen by constructing a phasor diagram containing \( V_{\text{eff}} \) and \( I_{\text{eff}} \) as shown in Fig. 10-10. If the phase current is resolved into two components, one in phase with the voltage, having a magnitude \( I_{\text{eff}} \cos (\theta - \phi) \), and one 90° out of phase with the voltage, with magnitude equal to \( I_{\text{eff}} \sin |\theta - \phi| \), then it is clear that the real power is given by the product of the magnitude of the voltage phasor and the component of the phasor current which is in phase with the voltage. Moreover, the product of the magnitude of the voltage phasor and the component of the phasor current which is 90° out of phase with the voltage is the reactive power \( Q \). It is common to speak of the component of a phasor which is 90° out of phase with some other phasor as a quadrature component. Thus \( Q \) is simply \( V_{\text{eff}} \) times the quadrature component of \( I_{\text{eff}} \). \( Q \) is also known as the quadrature power.

**Figure 10-10**

The current phasor \( I_{\text{eff}} \) is resolved into two components, one in phase with the voltage phasor \( V_{\text{eff}} \) and the other 90° out of phase with the voltage phasor. This latter component is called a quadrature component.

The sign of the reactive power characterizes the nature of a passive load in which \( V_{\text{eff}} \) and \( I_{\text{eff}} \) are specified. If the load is inductive, then \((\theta - \phi)\) is an angle between 0 and 90°, the sine of this angle is positive, and the reactive power is positive. A capacitive load results in a negative reactive power.

Just as a wattmeter reads the average real power drawn by a load, a varmeter will read the average reactive power \( Q \) drawn by the load. Both quantities may be metered simultaneously. In addition, watthourmeters and varhourmeters may be used simultaneously to record real and reactive energy used by any consumer during any desired time interval. From these records the average PF may be determined and the consumer's bill may be adjusted accordingly.

It is easy to show that the complex power delivered to several interconnected loads is the sum of the complex powers delivered to each of the individual loads, no matter how the loads are interconnected. For example, consider the two loads shown connected in parallel in Fig. 10-11. If rms values are assumed, the complex power drawn by the combined load is

\[
S = VI^* = V(I_1 + I_2)^* = V(I_1^* + I_2^*)
\]

**Figure 10-11**

A circuit used to show that the complex power drawn by two parallel loads is the sum of the complex powers drawn by the individual loads.
and thus

\[ S = V_1^* I_1^* + V_2^* I_2^* \]

stated.
These new ideas can be clarified by a practical numerical example.

**Example 10-7** Let us suppose that an industrial consumer is operating a 50-
W (67.1-hp) induction motor at a lagging PF of 0.8. The source voltage is 230
rms. In order to obtain lower electrical rates, the customer wishes to raise
PF to 0.95 lagging. Specify an arrangement by which this may be done.

**Solution:** Although the PF might be raised by increasing the real power
and maintaining the reactive power constant, this would not result in a
lower bill and is not a cure which interests the consumer. A purely reactive
load must be added to the system, and it is clear that it must be added in
parallel, since the supply voltage to the induction motor must not change.
The circuit of Fig. 10-11 is thus applicable if we interpret \( S_1 \) as the induction
motor's complex power and \( S_2 \) as the complex power drawn by the corrective
device.

The complex power supplied to the induction motor must have a real
part of 50 kW and an angle of \( \cos^{-1}(0.8) \), or 36.9°. Hence,

\[ S_1 = \frac{50 / 36.9°}{0.8} = 50 + j37.5 \text{ kVA} \]

In order to achieve a PF of 0.95, the total complex power must become

\[ S = \frac{50}{0.95 / \cos^{-1}(0.95)} = 50 + j16.43 \text{ kVA} \]

Thus, the complex power drawn by the corrective load is

\[ S_2 = -j21.07 \text{ kVA} \]

The necessary load impedance \( Z_2 \) may be found in several simple steps. We
select a phase angle of 0° for the voltage source, and therefore the current
drawn by \( Z_2 \) is

\[ I_2^* = \frac{S_2}{V} = -\frac{j21070}{230} = -j91.6 \text{ A} \]

or

\[ I_2 = j91.6 \text{ A} \]

Therefore,

\[ Z_2 = \frac{V_2}{I_2} = \frac{230}{j91.6} = -j2.51 \Omega \]

At the operating frequency of 60 Hz, this load can be provided by a 1056-µF
capacitor connected in parallel with the motor.

A capacitive load may also be simulated by a so-called synchronous capaci-
tor, a type of rotating machine. This is usually the more economical procedure
for small capacitive reactances (large capacitances). Whatever device is
decided, however, its initial cost, maintenance, and depreciation must be cov-
ered by the reduction in the electric bill.
Drill Problem 10-B. For the circuit shown in Fig. 10-12, find the complex power absorbed by the (a) 1-Ω resistor; (b) \(-j10\)-Ω capacitor; (c) 5 + j10-Ω impedance; (d) source.

\[\text{Ans: } 26.6 + j0 \text{ VA}; 0 - j1331 \text{ VA}; 532 + j1065 \text{ VA}; -559 + j266 \text{ VA}\]

Figure 10-12
See Drill Prob. 9.

Problems

1 A current source, \(i(t) = 2 \cos 500t\) A, a 50-Ω resistor, and a 25-\(\mu\)F capacitor are in parallel. Find the power being supplied by the source, being absorbed by the resistor, and being absorbed by the capacitor, all at \(t = \pi/2\) ms.

2 The current \(i = 2t^2 - 1\) A, \(1 \leq t \leq 3\) s, is flowing through a certain circuit element. (a) If the element is a 4-H inductor, what energy is delivered to it in the given time interval? (b) If the element is a 0.2-F capacitor with \(v(1) = 2\) V, what power is being delivered to it at \(t = 2\) s?

3 If \(v_c(0) = -2\) V and \(i(0) = 4\) A in the circuit of Fig. 10-13, find the power being absorbed by the capacitor at \(t\) equal to (a) 0; (b) 0.2 s; (c) 0.4 s.

Figure 10-13
See Prob. 3.

4 The circuit shown in Fig. 10-14 has reached steady-state conditions. Find the power being absorbed by each of the four circuit elements at \(t = 0.1\) s.

Figure 10-14
See Prob. 4.

5 Find the average power being absorbed by each of the five circuit elements shown in Fig. 10-15.

Figure 10-15
See Prob. 5.
the instantaneous power absorbed by the element is

\[ P(t) = i(t) \, v(t) \]

Let

\[ i(t) = I_m \cos(Wt + \phi_i) \]
\[ v(t) = V_m \cos(Wt + \phi_i) \]

\[ P(t) = \sqrt{\frac{V_m I_m}{2}} \, \cos(Wt + \phi_i) \, \cos(Wt + \phi_i) \]

\[ \frac{1}{T} \left[ \cos(\phi_i - \phi_e) + \cos(2Wt + \phi_i + \phi_e) \right] \]

\[ P_{av} = \frac{1}{T} \int_0^T P(t) \, dt = \frac{V_m I_m}{2} \cos(\phi_i - \phi_e) \left( \frac{\phi - \phi_e}{\theta} \right) \]

\[ P_{av} = \frac{1}{2} V_m I_m \cos \theta. \]
For a resistor \( \Phi_1 = \Phi_2 \implies \Theta = 0 \).

\[
P_{\text{av}} = \frac{1}{2} V I = \frac{1}{2} V I_R = \frac{1}{2} R I^2.
\]

For an impedance that is purely imaginary (reactive):

\[
\Theta = \Phi_1 - \Phi_2 = \pm 90^\circ.
\]

\[
P_{\text{av}} = \frac{1}{2} V I \cos(90^\circ) = 0.
\]

Purely reactive impedances, such as ideal inductors and capacitors, absorb zero power and are referred to as lossless elements.
\[
\begin{aligned}
\text{a) } & \quad C = \frac{1}{8} \text{ F} \\
V &= \frac{4}{6j + \frac{1}{j3(\frac{1}{8})} + 4} \\
&= \frac{40}{\sqrt{9 + \left(\frac{10}{3}\right)^2}} \quad \tan^{-1} \left(\frac{10/3}{1}\right) \\
&= 7.68 \angle -39.8^\circ \\
\frac{P}{4\pi} &= \frac{1}{2} |V|^2 / R = \frac{1}{2} \left(\frac{7.68}{4}\right)^2 = 7.37 \text{ W.}
\end{aligned}
\]

Note:
\[
I = \frac{10 L^0}{6j + \frac{1}{j3(\frac{1}{8})} + 4} = \frac{10}{4 + j10/3} = 1.92 \angle -39.8^\circ \text{ A}
\]

\[
\frac{P}{5 \text{ source}} = \frac{1}{2} V I \cos \theta = \frac{1}{2} \left(\frac{10}{1.92}\right) \cos (-39.8^\circ)
\]

= 7.37 \text{ W.}
Consider,

\[ Z_L = R_L + jX_L, \]

what is the circuit, it can be shown that the load absorbs \( P_{\text{max}} \) is

\[ Z_L = R_L + jX_L = R_T - jX_T = Z_T^* \]

where \( Z_T \) is the Thévenin equivalent impedance of the circuit. If \( X_T = 0 \) then

\[ Z_L = R_L = \sqrt{R_T^2 + X_T^2} \]

= \( 1/Z_T \).

\[ P_{\text{max}} = \frac{1}{8 R_T} \left( \frac{V_T}{2} \right)^2 \]
\[
\begin{align*}
4 \frac{L}{\omega} &= 4 \cdot \frac{18}{2\pi} \\
4 \frac{L}{\omega} &= 4\frac{L}{60^\circ} = 4L^{60^\circ} \\
8I_1 + (I_1 - I_2)(-j4) &= 4L^{60^\circ} \\
4I_2 + (I_2 - I_1)(-j4) + j8I_2 &= 0.
\end{align*}
\]

\[I_1 = 0.343 L^{-29^\circ} \quad I_2 = 0.242 L^{-164^\circ}\]

\[P_{8R} = \frac{1}{2} |I_1|^2 R = \frac{1}{2} \left(0.343\right)^2 (8) = 4.71 \text{ mw}.
\]

\[P_{4R} = \frac{1}{2} |I_2|^2 R = \frac{1}{2} \left(0.242\right)^2 (4) = 17.6 \text{ mw}.
\]
b) From \( \frac{4\pi}{2\pi} \Rightarrow \) \[ z_T = \frac{8}{5} + j \frac{24}{5} \]

\[ V_T = \frac{4}{\sqrt{3}} \angle -123.43^\circ \]

\[ z_L = z_T^* = \frac{8}{5} - j \frac{24}{5} \]

\[ P_{max} = \frac{1}{8 R_T} \left| V_T \right|^2 = \frac{(4/\sqrt{3})^2}{8 \left( \frac{8}{5} \right)} = 2.5 \text{ W} \]
\[ R_L = \sqrt{R_T^2 + X_T^2} = \sqrt{\left(\frac{8}{5}\right)^2 + \left(\frac{24}{5}\right)^2} = 5.06 \, \Omega. \]

\[ P_{L_{\text{max}}} = \frac{1}{2} \left| \frac{R_L}{Z_T + R_L} \right| V_T^2 / R_L \]

\[ = \frac{1}{2} \frac{1}{\left(\frac{8}{5} + 5.06\right)^2 + \left(\frac{24}{5}\right)^2} \left(\frac{4}{5}\right)^2 (5.06) \]

\[ \left(\frac{8}{5} + 5.06\right)^2 + \left(\frac{24}{5}\right)^2 \]

\[ = 12 \, \text{W} \]
Recall: If \( i(t) = I \cos (\omega t + \phi) \), then

\[
I_e = \frac{I_{\text{rms}}}{\sqrt{2}} = \frac{I}{\sqrt{2}} \approx 0.707 I
\]

\[
P_R = \frac{1}{2} VI = \frac{V}{\sqrt{2}} \frac{I}{\sqrt{2}} = V_e I_e = I_e^2 R = \frac{V_e^2}{R}
\]

and

\[
P_x = \frac{1}{2} VI \cos \theta = \frac{V}{\sqrt{2}} \frac{I}{\sqrt{2}} \cos \theta = V_e I_e \cos \theta.
\]
Consider an arbitrary load.

We say \( Z \) is inductive if \( x > 0 \) and capacitive if \( x < 0 \).

\[
Z = R + jx
\]

where

\[
V(t) = V \cos(wt + \phi_1)
\]

\[
i(t) = I \cos(wt + \phi_2)
\]

then

\[
P_{av,\text{Load}} = \frac{1}{2} VI \cos(\phi_1 - \phi_2) = \frac{V I}{\cos \Theta} \quad \text{apparent power (VA)}
\]

\[
\text{avg power} = \text{(apparent power)} \times \text{(power factor)}
\]

\[
Z = \frac{v}{i} = \frac{V / \epsilon_1}{I / \epsilon_2} = \frac{V}{I} \cot \theta \quad \Rightarrow \quad \text{ang } Z = \theta
\]

Note that \( \theta = \tan^{-1} \left( \frac{X}{R} \right) \)

\[
X > 0 \quad \Rightarrow \quad \theta > 0 \quad (\phi_1 > \phi_2)
\]

\[
X < 0 \quad \Rightarrow \quad \theta < 0 \quad (\phi_1 < \phi_2)
\]

Again, pf.
We say, if current lags the voltage, the result is a lagging Pf; and if it leads the voltage, the result is a leading Pf.
\[ w = (60 \text{ Hz}) (2\pi) \]

\[ Z = (8 + j 2w) \parallel (-j \frac{8}{w}) = 0.021 \angle -90^\circ \]

\[ \sin \theta = 0.021 \]
\[ \theta = -90^\circ, \quad \text{pf} = \cos 90^\circ = 0. \]

b) Since \( Z = \frac{V}{I} \) has a negative angle, the current leads the voltage, so the \text{pf} is \textit{leading}.

c) \[ \cos^{-1}(0.8) = 36.87^\circ \]
\[ \angle Z = 36.87^\circ \]
\[ \left| \frac{(8 + j 2w) \parallel (-j \frac{1}{w})}{w = 120^\circ} \right| = 36.87^\circ \quad \Rightarrow \quad C = 3.49 \text{ MF} \]
\[ \frac{4.32}{227} : \quad Pf = 0.95 \]

\[ P_{av} = \frac{I_e V_e \cos \theta}{\sqrt{\frac{115}{95}}} = 0 \quad I_e = 4.57 \text{ A (r.m.s)} \]

\[ \frac{4.33}{228} : \]

\[ I = V - \sqrt{I_1 I_2} \]

\[ P_{av_1} = 5500 \text{ W, } \quad Pf_1 = 0.8 \]

\[ P_{av_2} = 1600 \text{ W, } \quad Pf_2 = 0.9 \]

\[ I_{e1} = \frac{500}{(230)(0.8)} = 2.71 \text{ A (r.m.s)} = 2.71 \angle -36^{\circ} \]

\[ I_{e2} = \frac{1440}{(230)(0.9)} = 4.83 \text{ A (r.m.s)} = 4.83 \angle -25.8^{\circ} \]

a) \[ I = I_{e1} + I_{e2} = 2.71 \angle -16.9^{\circ} + 4.87 \angle -25.8^{\circ} = 7.52 \angle -29.8^{\circ} \text{ A} \]

b) \[ \varphi = 120^{\circ} \times 0.87 = 100.8^{\circ} \]
Complex power:

\[ P_{av} = V_e I_e \cos \theta = V_e I_e \{Re[e^{j\theta}]\} \]

\[ = V_e I_e \left\{ Re\left[e^{j(\phi - \phi_2)}\right]\right\} \]

\[ = V_e I_e \left\{ Re\left[e^{j\phi} \cdot e^{-j\phi_2}\right]\right\} \]

\[ = Re\left\{ \left(V_e e^{j\phi_1}\right) \left(I_e e^{-j\phi_2}\right)\right\} \]

\[ = Re\left\{ \left(V_rms I_{rms}\right)^* \left(I_rms I_{rms}\right)^*\right\} \]

\[ P_{av} = Re\left\{ V_{rms} I_{rms}^* \right\} \]

Let

\[ P = V_{rms} I_{rms}^* = V_e I_e e^{j\theta} \]

Complex power

\[ = \frac{V_e I_e \cos \theta + j \frac{V_e I_e \sin \theta}{P_{av}}}{P_{av}} \]

\[ = \frac{V_e I_e \cos \theta + \frac{V_e I_e \sin \theta}{P_{av}}}{P_{av}} \]

\[ Q = \text{Reactive power (VAR)} \]

\[ \therefore P = P_{av} + jQ \]
\[ P_1 = \sqrt{P_{av}^2 + Q^2} = V_e I_e \quad \text{apparent power} \]

\[ \theta = \phi \quad \text{power factor angle}. \]

The power triangle is shown below:

\[ \sqrt{P_{av}^2 + Q^2} \quad \Rightarrow \quad \text{Pf.} = \cos \theta = \frac{P_{av}}{\sqrt{P_{av}^2 + Q^2}} \]
From 4.18 \Rightarrow I_1 = \sqrt{\frac{2}{17}} \angle -29^\circ \\

I_2 = \frac{1}{\sqrt{17}} \angle -164^\circ \Rightarrow I_2^* = \frac{1}{\sqrt{17}} \angle 29^\circ \\

a) P_S = \frac{1}{2} \frac{V_T}{S} I_2^* = \frac{1}{2} \cdot (4 \angle -60^\circ) \cdot (\frac{1}{\sqrt{17}} \angle 29^\circ) \\
= 0.686 \angle -31^\circ \text{ VA} \\

b) |P_S| = 0.686 \text{ VA} \\

c) \theta = -31^\circ \Rightarrow \cos \theta = 0.86 \text{ leading}