Chapter 7

Inductance and Capacitance

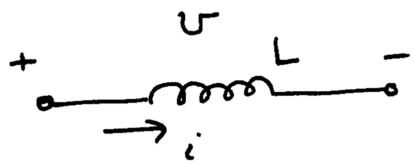
Inductors are circuit elements based on phenomena associated with magnetic fields. The source of the magnetic field is charge in motion, or current. If the current is varying with time, the magnetic field is varying with time. A time-varying magnetic field induces a voltage in any conductor that is linked by the field. The circuit parameter of inductance relates the induced voltage to the current.

Capacitors are circuit elements based on phenomena associated with electric fields. The source of the electric field is separation of charge, or voltage. If the voltage is varying with time, the electric field is varying with time. A time-varying electric field produces a displacement current in the space occupied by the field. The circuit parameter of capacitance relates the displacement current to the voltage.

Remarks

Energy can be stored in both magnetic and electric fields. Therefore, inductors and capacitors are passive elements that are capable of storing energy.

the inductor



$$v = L\,\frac{di}{dt}$$

$$v\,dt = L\left(\frac{di}{\cancel{dt}}\right)\cancel{dt} = L\,di$$

$$\int_{t_0}^{t} v\,d\tau = L\int_{i(t_0)}^{i(t)} dx = L\left(i(t) - i(t_0)\right)$$

$$\therefore\quad i(t) = \frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0)\,.$$

$$P = \underset{L\frac{di}{dt}}{\cancel{v}}\,i = v\left[\frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0)\right]$$

Also, $\quad P = vi = L \dfrac{di}{dt} i$

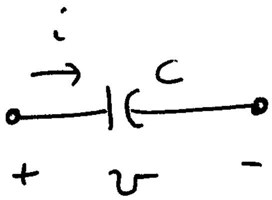
$$P = \frac{dw}{dt} = L i \frac{di}{dt}$$

$$\therefore \quad dw = L i \, di$$

$$\int_0^W dx = L \int_0^i y \, dy$$

$$\therefore \quad W = \frac{1}{2} L i^2 .$$

the Capacitor



$$i = C\,\frac{dv}{dt}$$

$$i\,dt = C\,dv \quad \Rightarrow \quad \frac{1}{C}\,i\,dt = dv$$

$$\frac{1}{C}\int_{t_0}^{t} i\,d\tau = \int_{v(t_0)}^{v(t)} dx = v(t) - v(t_0)$$

$$\therefore\; v(t) = \frac{1}{C}\int_{t_0}^{t} i\,d\tau + v(t_0).$$

$$P = v\,\underbrace{\cancel{i}}_{C\frac{dv}{dt}} = C\,v\,\frac{dv}{dt}$$

or

$$\therefore \quad p = i\left[\frac{1}{c}\int_{t_0}^{t} i\, d\tau + v(t_0)\right].$$

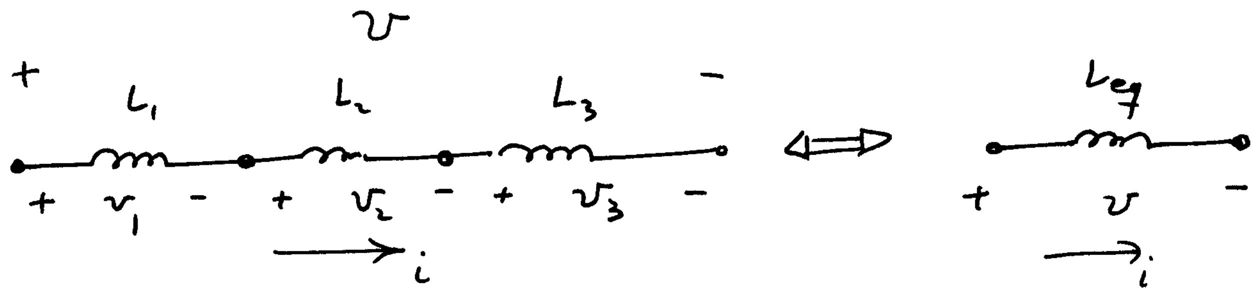
$$p = \frac{dw}{dt} \Longrightarrow dw = p\,dt \qquad (1)$$

Also, $$p = c\,v\,\frac{dv}{dt} \Longrightarrow p\,dt = c\,v\,dv. \qquad (2)$$

$$(2) \div (1) \Longrightarrow$$

$$\int_0^W dw = c\int_0^v \cancel{v}_y\, d\cancel{v}_y \Longrightarrow W = \frac{1}{2}c\,v^2 \;/\!/.$$

Series - Parallel Combinations of L & C.



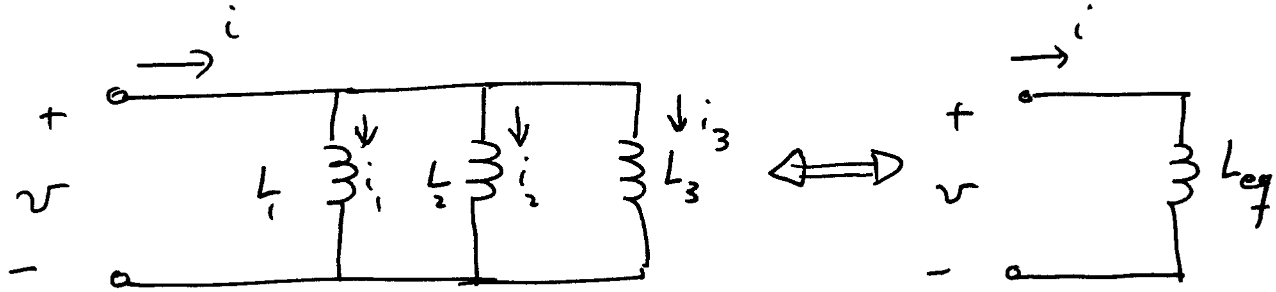
$$v_1 = L_1 \frac{di}{dt}$$

$$v_2 = L_2 \frac{di}{dt}$$

$$v_3 = L_3 \frac{di}{dt}$$

$$\underbrace{v}_{L_{eq}\frac{di}{dt}} = v_1 + v_2 + v_3 = (L_1 + L_2 + L_3)\frac{di}{dt}$$

$$\therefore \quad L_{eq} = L_1 + L_2 + L_3 . \quad //$$



$$i_1 = \frac{1}{L_1}\int_{t_0}^{t} v\,d\tau + i_1(t_0)$$

$$i_2 = \frac{1}{L_2}\int_{t_0}^{t} v\,d\tau + i_2(t_0)$$

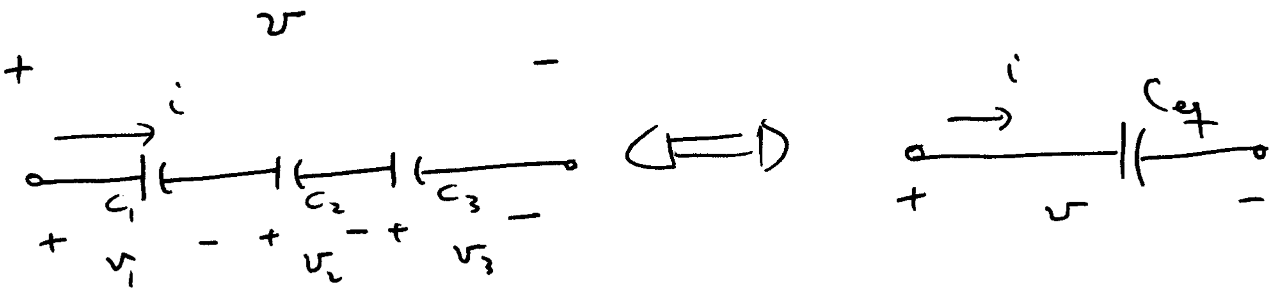
$$i_3 = \frac{1}{L_3}\int_{t_0}^{t} v\,d\tau + i_3(t_0)$$

$$i = i_1 + i_2 + i_3 = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int_{t_0}^{t} v\,d\tau + \left(i_1(t_0) + i_2(t_0) + i_3(t_0)\right).$$

$$i = \frac{1}{L_{eq}}\int_{t_0}^{t} v\,d\tau + i(t_0).$$

$$\therefore \quad \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \quad ; \qquad i(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0).$$

//



$$v = v_1 + v_2 + v_3$$

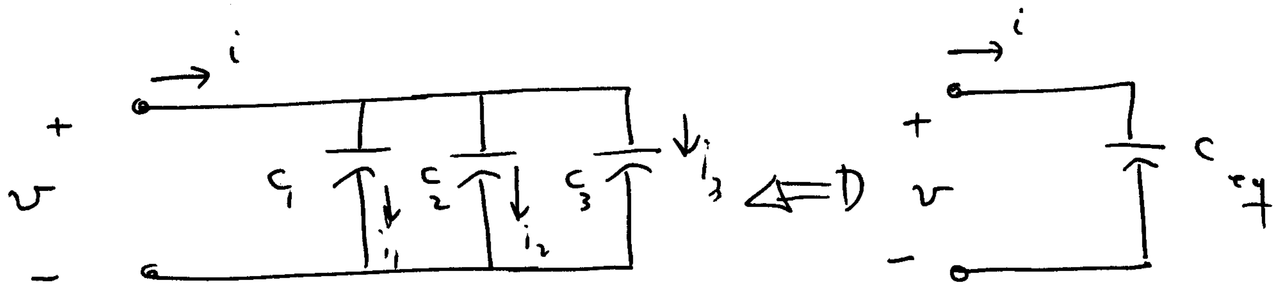
$$v_i = \frac{1}{C_i}\int_{t_0}^{t} i \, d\tau + v_i(t_0) \quad ; \quad i = 1, 2, 3.$$

$$\therefore \; v = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)\int_{t_0}^{t} i \, d\tau + \left(v_1(t_0) + v_2(t_0) + v_3(t_0)\right)$$

$$= \frac{1}{C_{eq}}\int_{t_0}^{t} i \, d\tau + v_{eq}(t_0)$$

$$\therefore \; \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \quad //$$

$$v_{eq}(t_0) = v_1(t_0) + v_2(t_0) + v_3(t_0). \quad //.$$



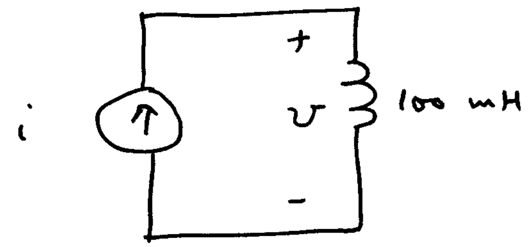
$$i = i_1 + i_2 + i_3$$

$$= C_1 \frac{dv}{dt} + C_2 \frac{dv}{dt} + C_3 \frac{dv}{dt}$$

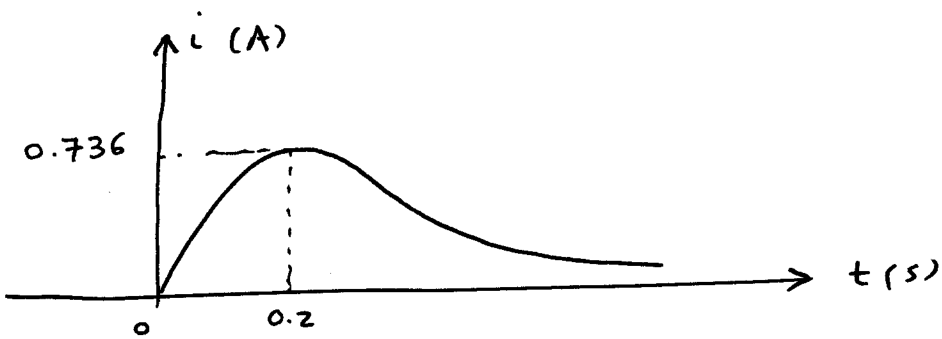
$$i = C_{eq} \frac{dv}{dt}$$

$$\therefore \quad C_{eq} = C_1 + C_2 + C_3 \quad //$$

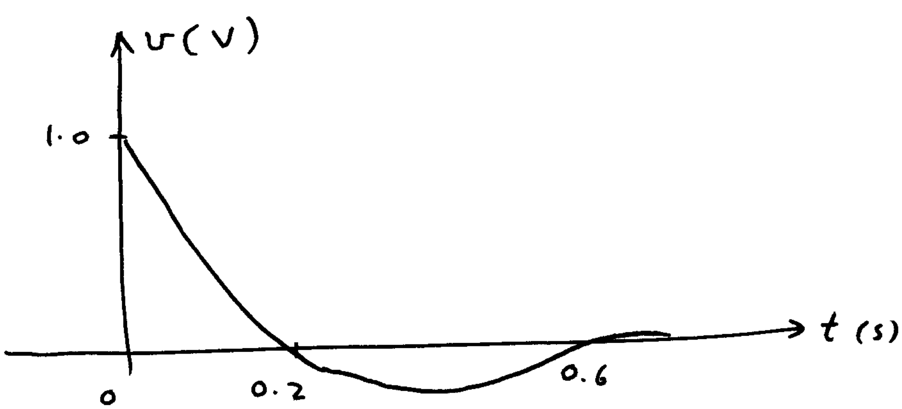
Exa $\frac{6.1}{215}$:



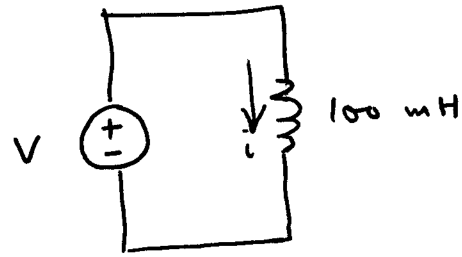
$$i = 10\, t\, e^{-5t} \quad u(t).$$



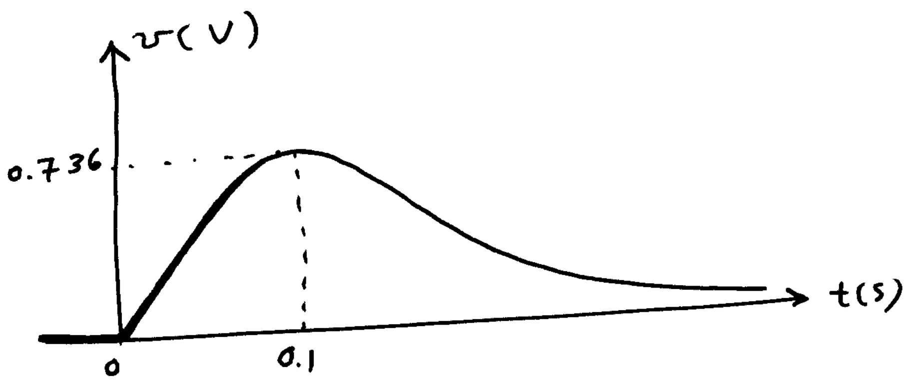
$$v = \overset{0.1}{\cancel{L}} \frac{di}{dt} = (0.1) \frac{d}{dt}\left(10\, t\, e^{-5t}\right) = e^{-5t}\,(1-5t) \quad u(t).$$



Exa $\frac{6.2}{217}$:

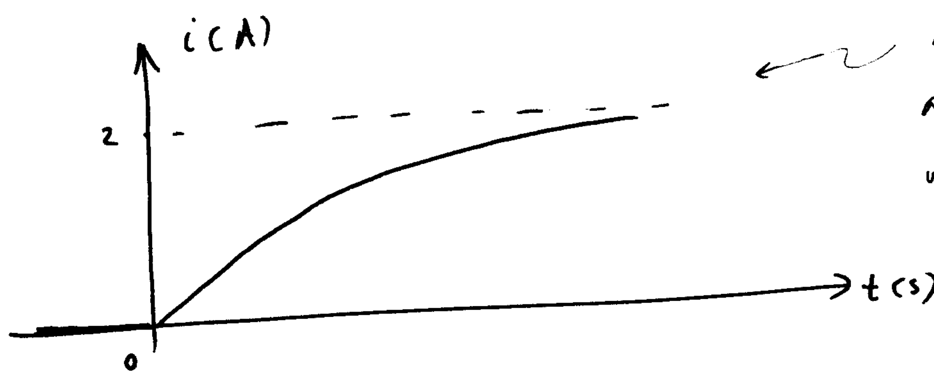


$$v(t) = 20\,t\,e^{-10t}\,u(t)$$



$$i(t) = \frac{1}{\underset{0.1}{\cancel{L}}}\int_0^t \underset{20\tau e^{-10\tau}}{\cancel{v(\tau)}}\,d\tau + \overset{0}{\cancel{i(0)}} = 2\left(1 - 10t\,e^{-10t} - e^{-10t}\right) A$$

$t \geq 0$



note that $i \to 2$, a constant current. we say more about this after discussing the energy stored in L!

(see next page) →

$$v(t) = 20t\, e^{-10t}\, u(t)$$

$$i(t) = 2(1 - 10t\, e^{-10t} - e^{-10t})\, u(t)$$

$$p(t) = v(t)\, i(t)$$

$$w(t) = \int p(t)\, dt$$

- Note that $p > 0$; $\forall t$, and hence energy always is being <u>stored</u> during the duration of the v pulse.

- Because L is ideal, the stored energy cannot dissipate after $v \to 0$. Therefore, a sustained current circulates in the circuit.

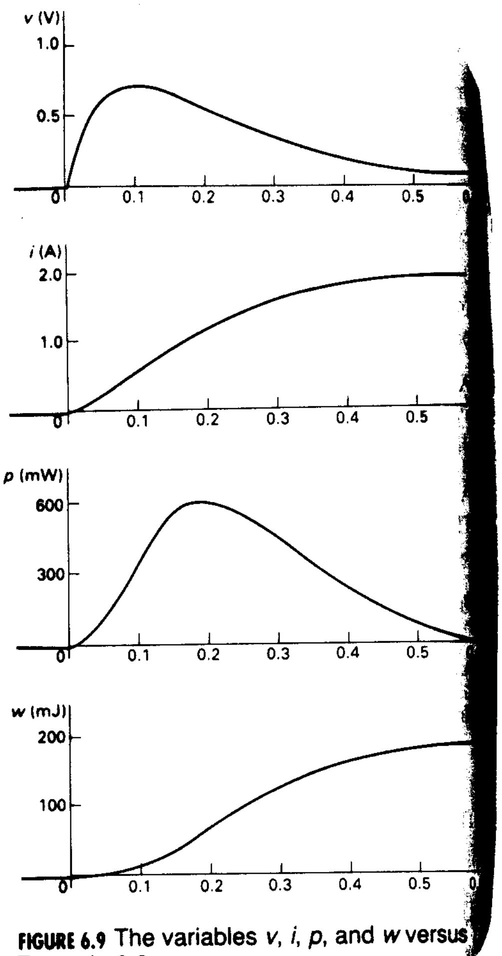
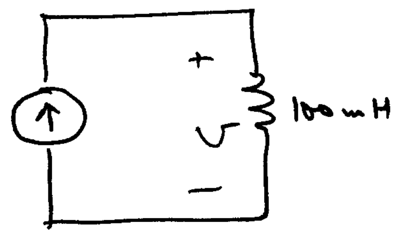


FIGURE 6.9 The variables v, i, p, and w versus Example 6.2.

Exa $\frac{6.3}{218}$:

$i = 10\,t\,e^{-5t}\,u(t)$



$v = e^{-5t}(1-5t)\,u(t)$

$p = vi \longrightarrow$

$W = \int p\,dt \longrightarrow$

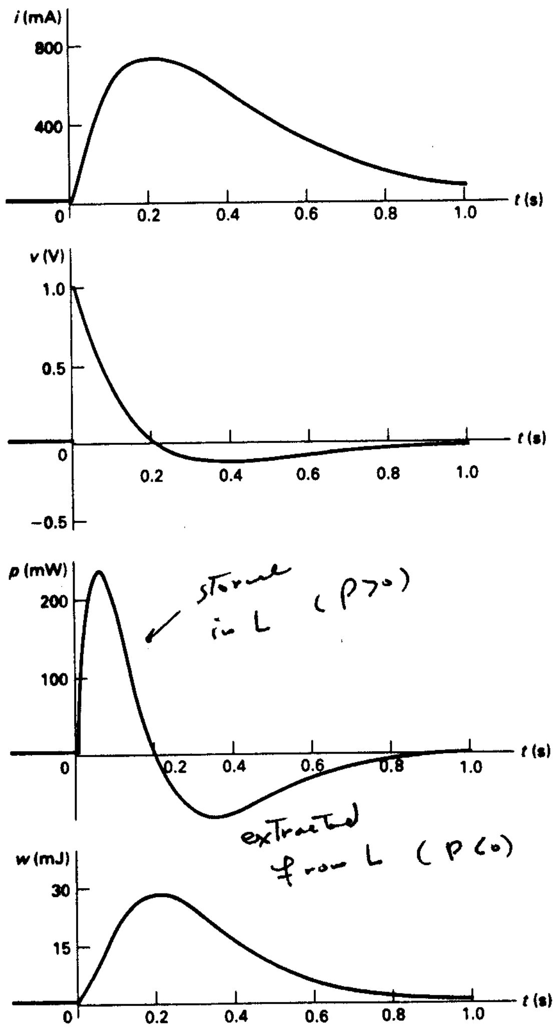
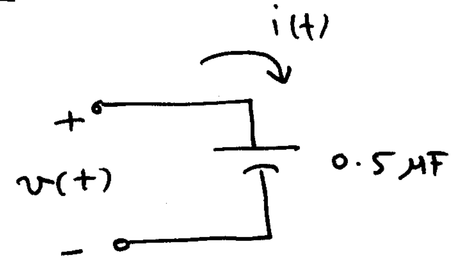


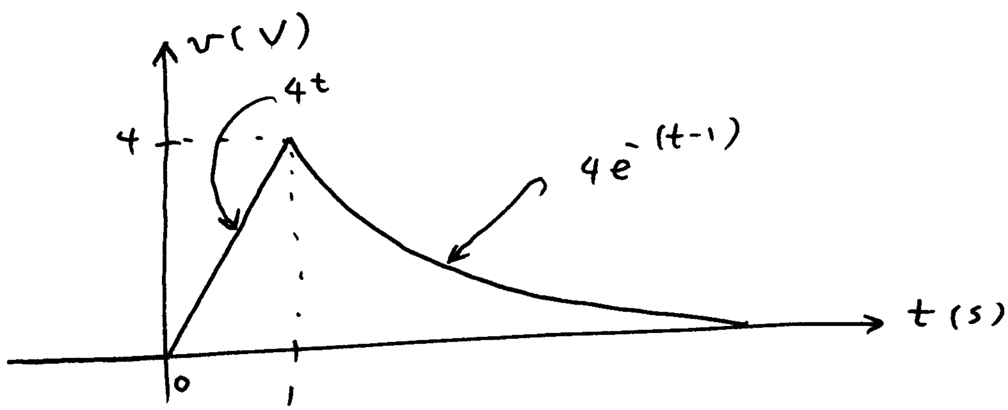
FIGURE 6.8 The variables *i*, *v*, *p*, and *w* versus *t* for Example 6.1.

Note that $p>0$ $(0 \le t \le 0.2)$ represents the energy stored in L; and $p<0$ $(0.2 < t < \infty)$ shows the energy extracted.

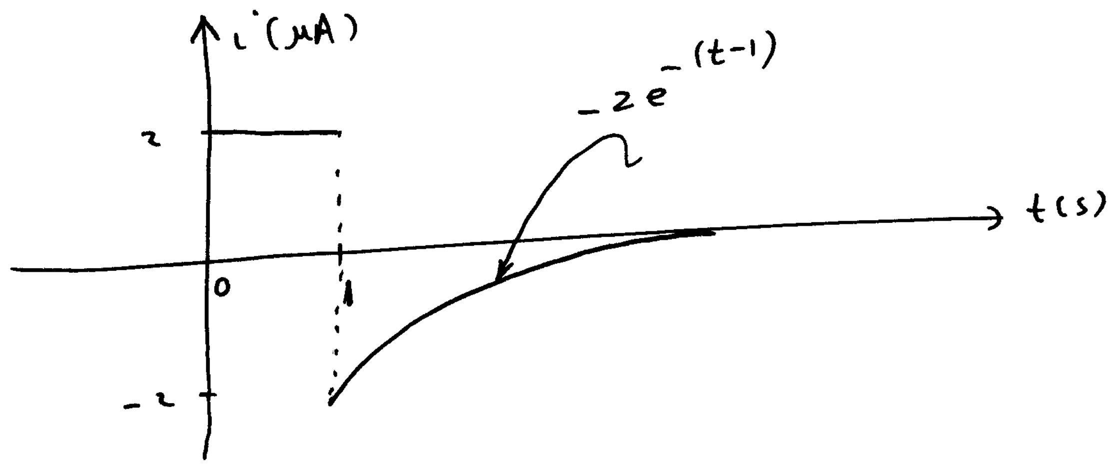
Exa $\frac{6.4}{223}$



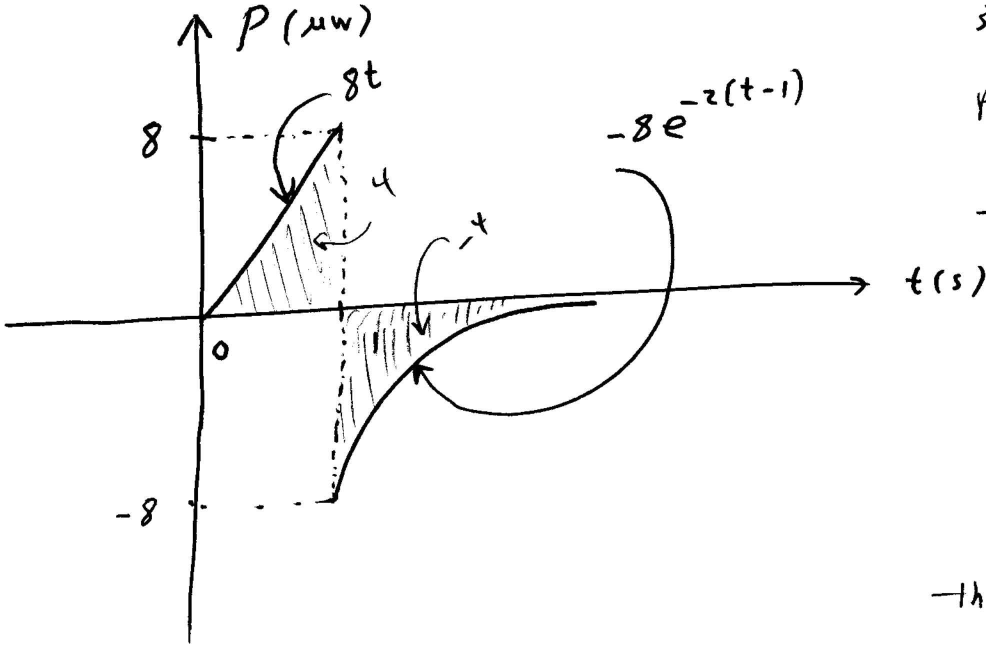
$$v(t) = \begin{cases} 0, & t \le 0 \\ 4t, & 0 \le t \le 1 \\ 4(e^{-(t-1)}), & 1 \le t \le \infty \end{cases}$$



$$i(t) = C\frac{dv}{dt} = (0.5\,\mu F)\frac{dv}{dt}$$



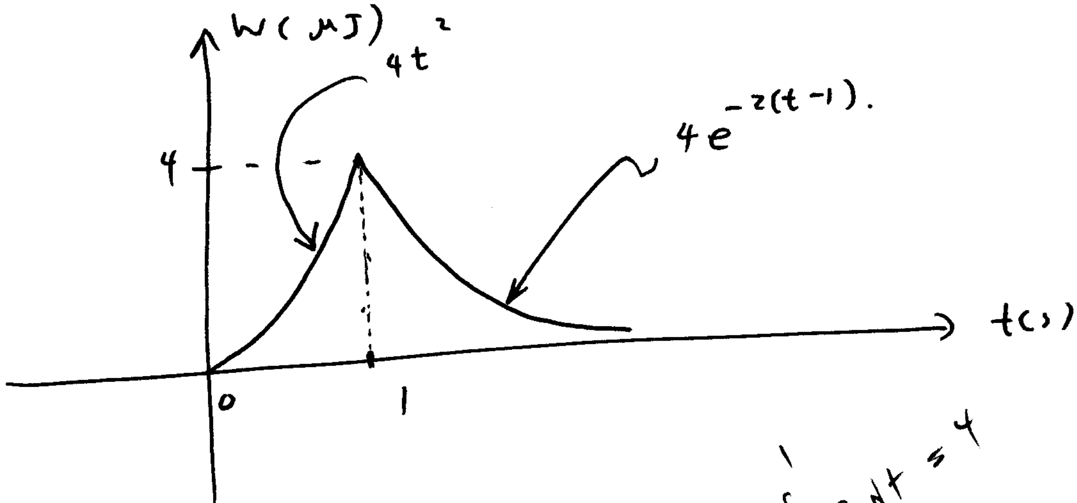
$p = v\,i$



- energy is being stored in C whenever $p > 0$; i.e.; $0 \le t \le 1$.
- energy is being delivered by C whenever $p < 0$; i.e.; $1 < t$.

this is also seen from the graph of w below.

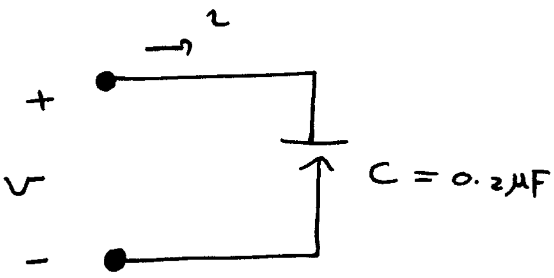
$w = \frac{1}{2} C v^2$

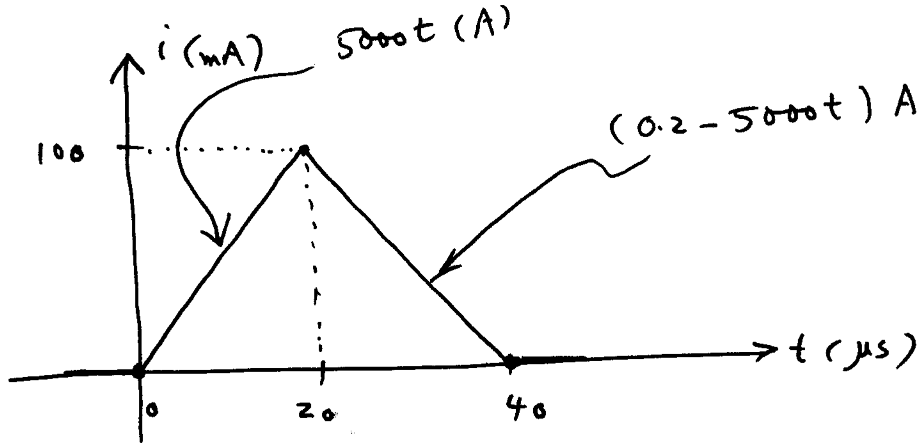


Note: $\int_0^1 p\,dt = 4$

$\int_1^\infty p\,dt = -4$

Exa $\frac{6.5}{224}$





$v(t) = 0 \; ; \qquad t \le 0$

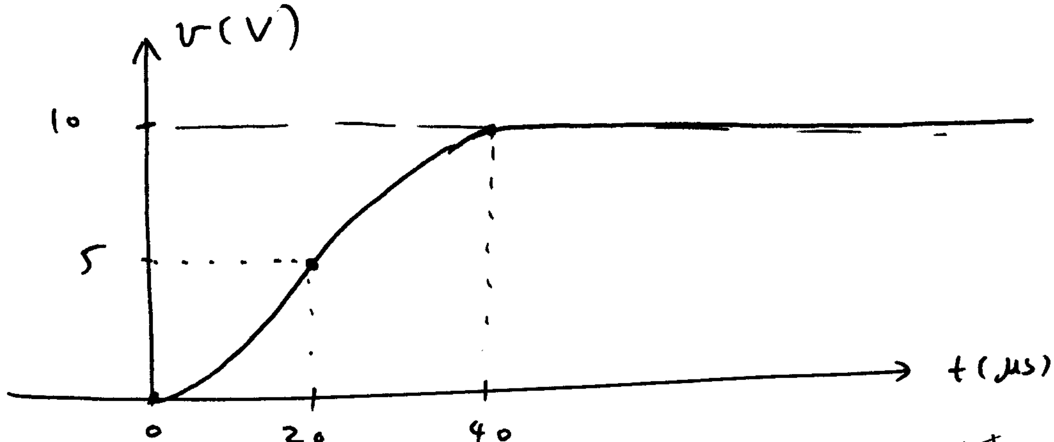
$$v(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau + v(0) \; ; \qquad 0 \le t \le 20 \; \mu s$$

$$= \frac{1}{(0.2)(10^{-6})}\int_0^t (5000\,\tau)\,d\tau + 0 = \underbrace{12.5\times10^{9}\,t^2}\; V \; ;$$

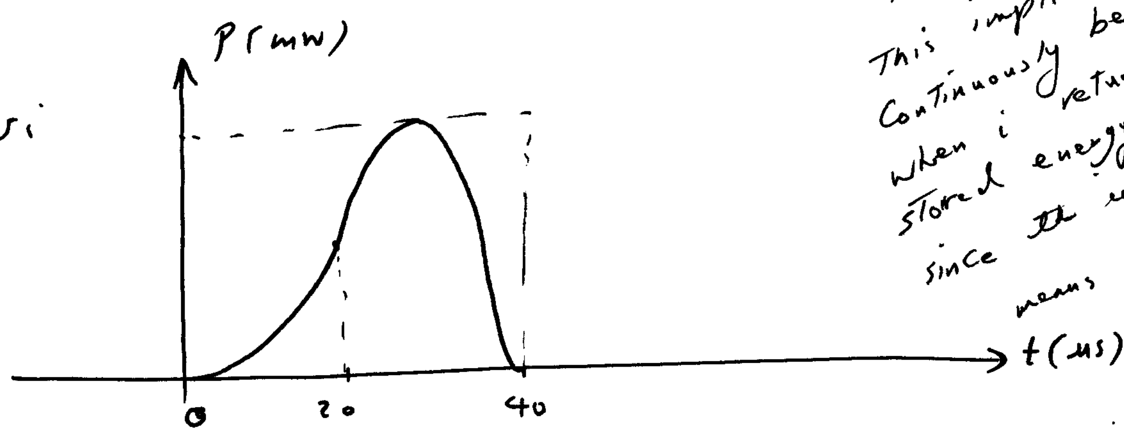
$$v(t) = \frac{1}{C}\int_{20\mu s}^{t} i(\tau)\,d\tau + v(20\,\mu s) \; ; \qquad 20\,\mu s \le t \le 40\,\mu s$$

$$= \frac{1}{(0.2)(10^{-6})}\int_{20\mu s}^{t} (0.2 - 5000\,\tau)\,d\tau + 5 = (10^6 t - 12.5\times10^{9} t^2 - 10)\; V$$

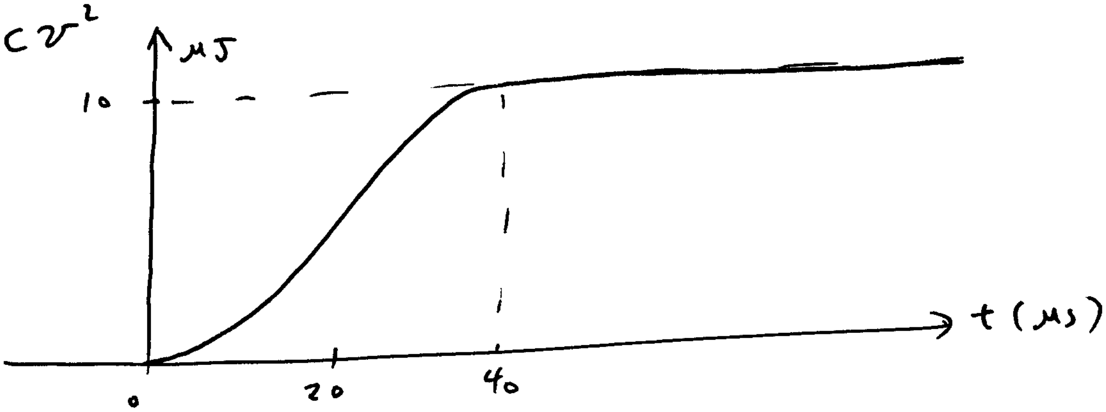
$$v(t) = \frac{1}{C}\int_{40\,\mu s}^{t\,\cancel{\infty}} \cancel{i(\tau)\,d\tau}^{\;0} + \cancel{v(40\,\mu s)}_{\;10\,V}, \qquad t \geq 40\,\mu s$$



$P = vi$



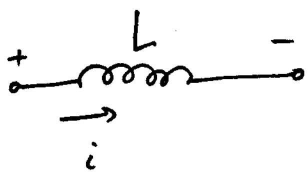
$W = \frac{1}{2} C v^2$



Note that $p > 0$; $\forall t$.
This implies that "energy" is continuously being stored in C. When i returns to zero, the stored energy is trapped since the ideal C offers no means for dissipating energy.

$\therefore$ The voltage remains on C after i returns to zero.

the inductor : (Review)



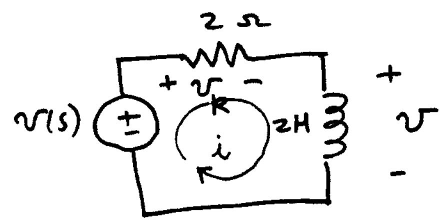
$$v = L\frac{di}{dt}$$

$$W_L(t) = \frac{1}{2} L\, i^2(t)$$

$$W_L(t) = \int v(t)\, i(t)\, dt$$

$$= \int L \frac{di}{\cancel{dt}}\, i(t)\, \cancel{dt} = \frac{1}{2} L\, i^2(t)$$

#3.1 / 156 :



$$v = L\frac{di}{dt} = 2\frac{di}{dt}$$

$$W_L = \frac{1}{2} L\, i^2(t) = i^2(t)$$

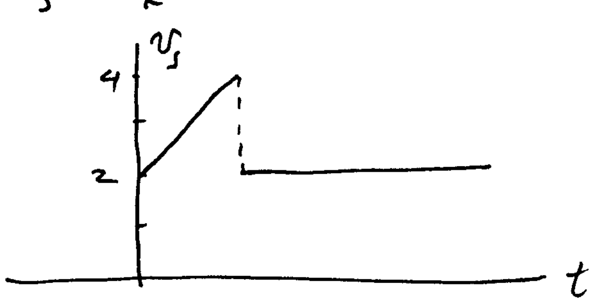
$$P_R(t) = \cancel{v_R}\, i = i^2 \cancel{R}\, 2$$

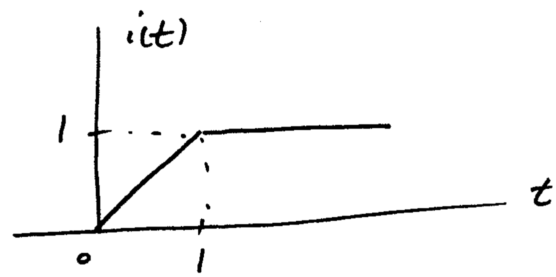
iR

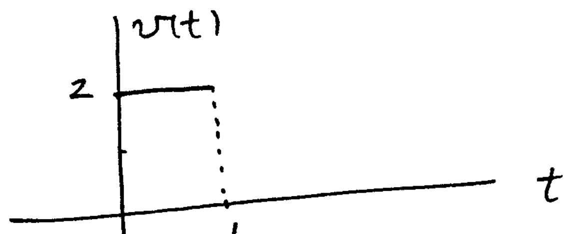
$$v_R(t) = iR = 2i$$

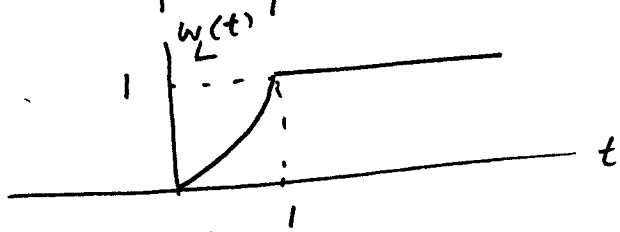
$$-v_s + v_R + v = 0$$

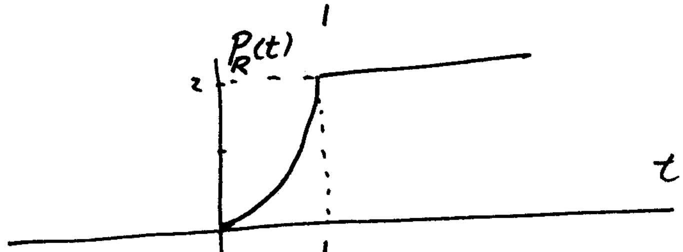
$$v_s = v_R + v$$

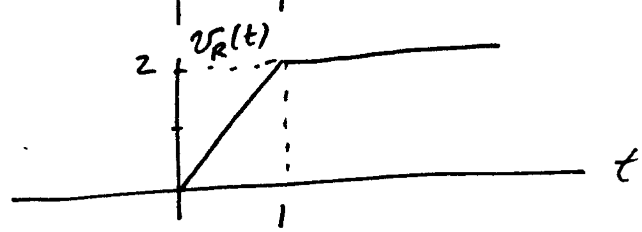












\#3.6
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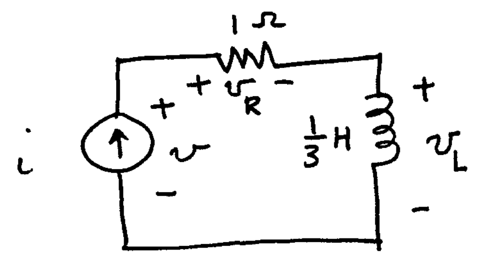
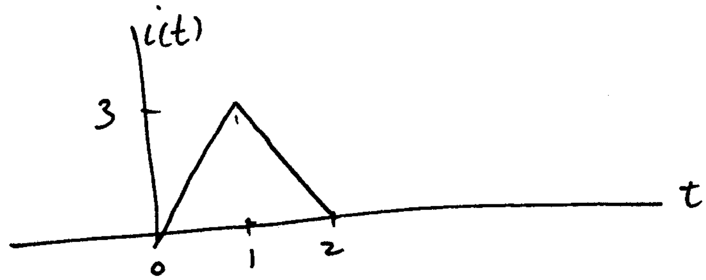
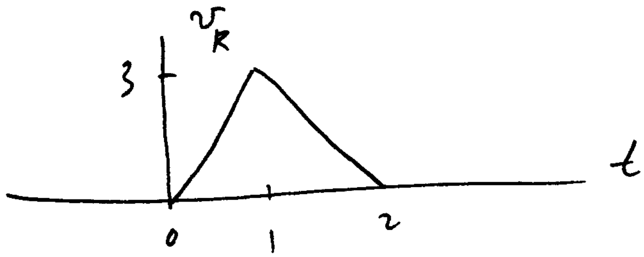


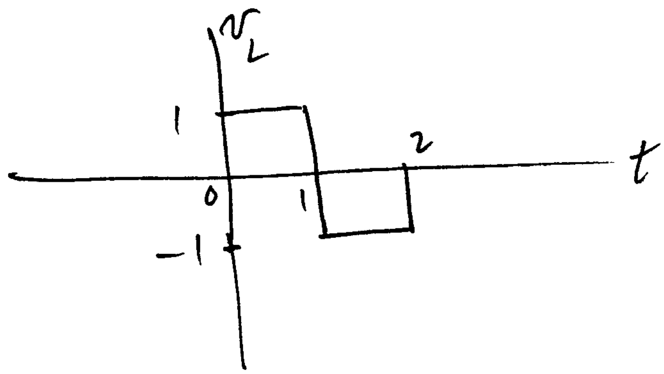
$$v_R = iR = i$$

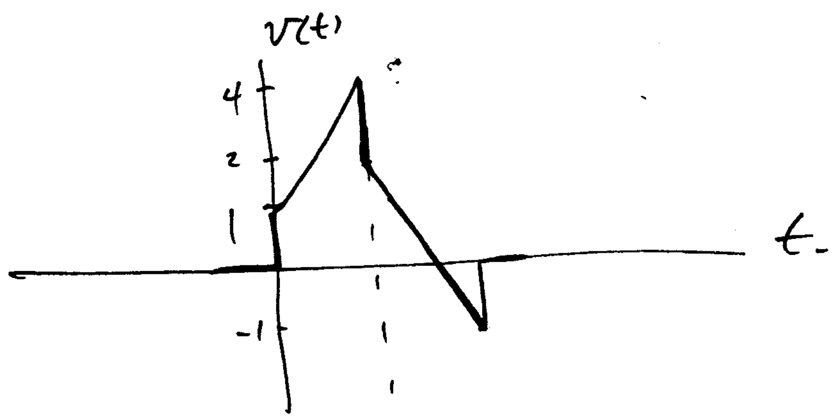
$$v_L = L\frac{di}{dt} = \frac{1}{3}\frac{di}{dt}$$

$$v = v_R + v_L$$

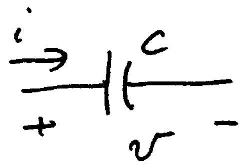








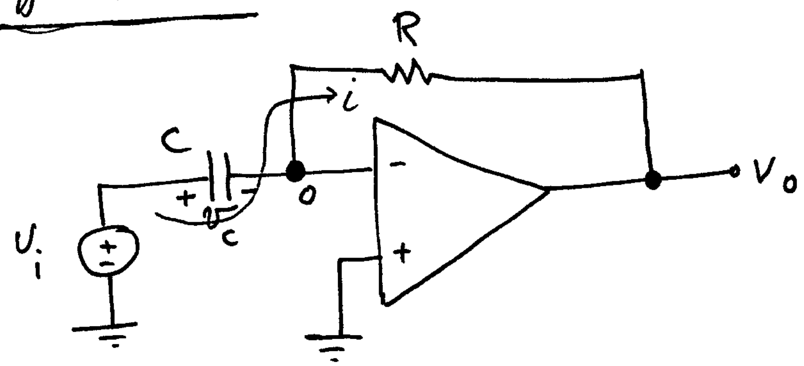
the Capacitor (Review)



$$i = C\frac{dv}{dt}$$

$$W_C(t) = \frac{1}{2} C v^2(t) .$$

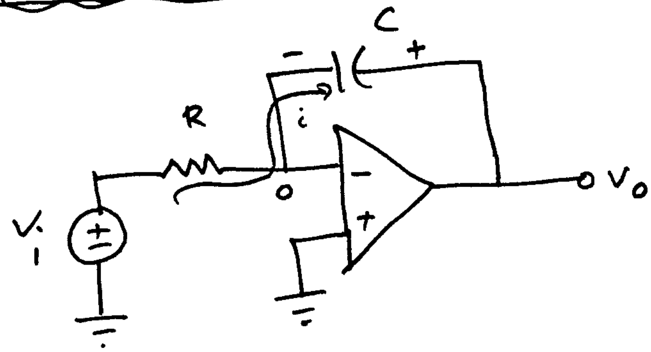
differentiator :



$$C\frac{dv_c}{dt} = i = \frac{0 - V_o}{R} , \quad V_C = V_i$$

$$\therefore \quad V_o = - RC\frac{dV_i}{dt}$$

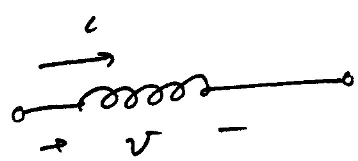
Integrator (# $\frac{3.22}{161}$)



$$\frac{V_i}{R} \; \cancel{i} = -C\,\frac{d\cancel{v_c}^{\,V_o}}{dt} \Rightarrow V_o = -\frac{1}{RC}\int V_i\,dt$$

Important Remark :

Consider an inductor



$$i = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt$$

Then at $t = a \Rightarrow i(a) = \frac{1}{L}\int_{-\infty}^{a} v(t)\,dt$

$$i(a+\varepsilon) = \frac{1}{L}\int_{-\infty}^{a+\varepsilon} v(t)\,dt = \underbrace{\frac{1}{L}\int_{-\infty}^{a} v(t)\,dt}_{i(a)} + \frac{1}{L}\int_{a}^{a+\varepsilon} v(t)\,dt$$

if $\varepsilon \to 0 \Rightarrow i(a+\varepsilon) = i(a)$.

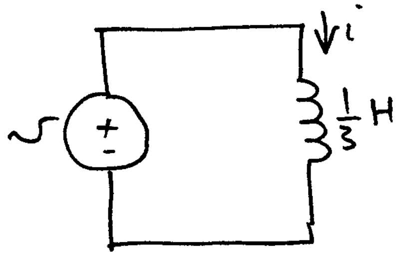
$\therefore$ the current through an inductor cannot change instantaneously. This fact can be described as follows too.

$$W_L = \frac{1}{2} L i^2(t) \quad \text{and} \quad P_L = \frac{d}{dt} W_L$$

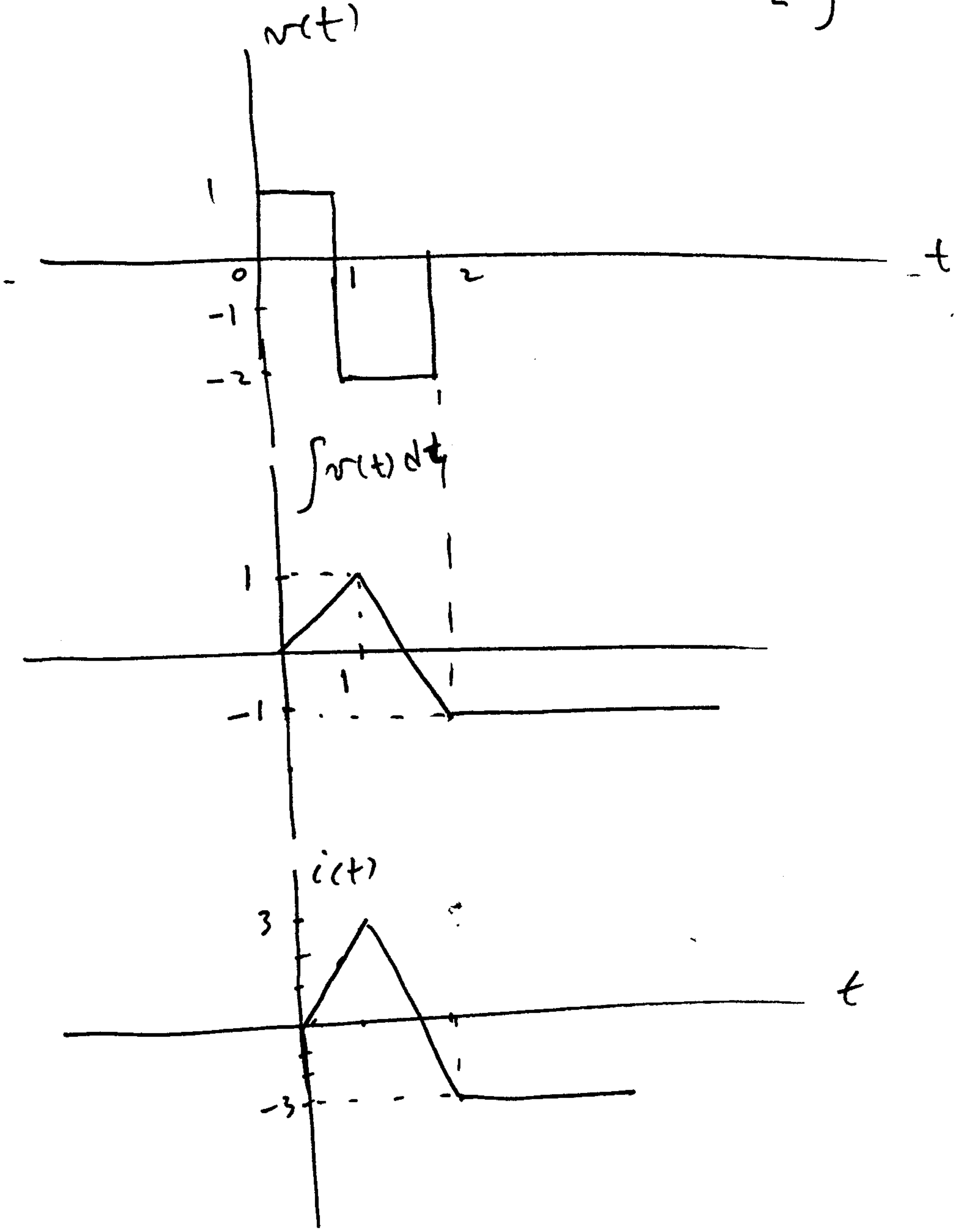
Note that a rapid change in $i(t)$ requires an infinite power which is not physically realizable.

In a similar manner, it can be shown that the voltage across a capacitor cannot change instantaneously.

$\frac{3.17}{159}$:



$$i = \frac{1}{L}\int v(t)\,dt$$

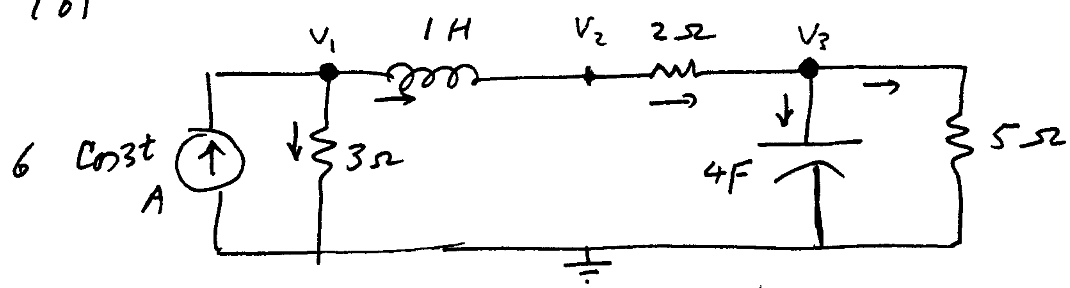


Important Circuit Concepts

Just as we analyzed circuits with R's and energy sources, so can we write a set of node or mesh eq^{n}s for circuits that contain L's and C's.

Exa.

3.23 : 161

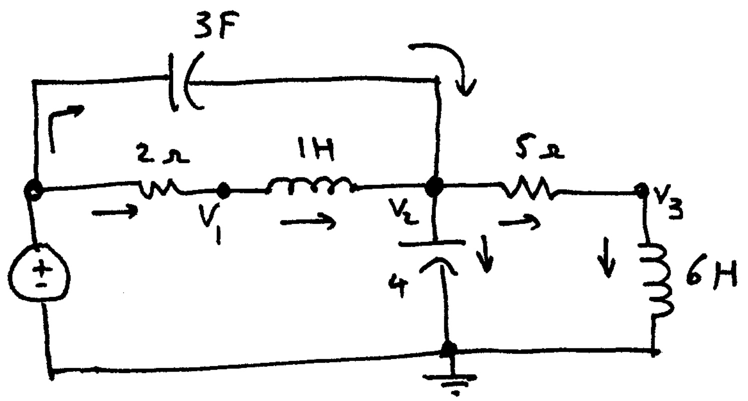


$$6 \cos 3t = \frac{V_1}{3} + \int (V_1 - V_2)\, dt$$

$$\int (V_1 - V_2)\, dt = \frac{V_2 - V_3}{2}$$

$$\frac{V_2 - V_3}{2} = \overset{4}{\cancel{1}} \frac{dV_3}{dt} + \frac{V_3}{5}$$

$\frac{3.24}{161}$:

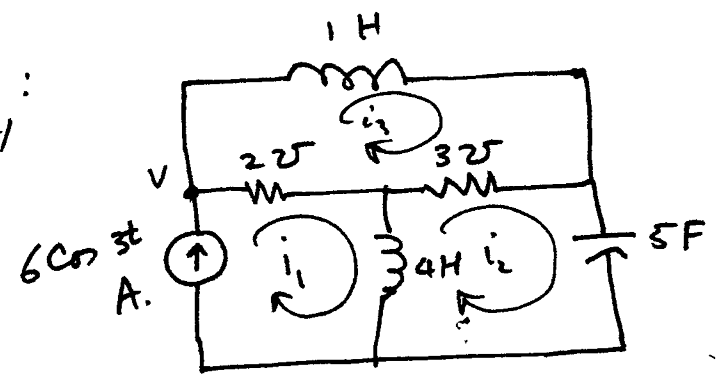


$$\frac{7\sin 4t - v_1}{2} = \int (v_1 - v_2)\,dt$$

$$\int (v_1 - v_2)\,dt + 3\,\frac{d\,(7\sin 4t - v_2)}{dt} = 4\,\frac{dV_2}{dt} + \frac{V_2 - V_3}{5}$$

$$\frac{V_2 - V_3}{5} = \frac{1}{6}\int V_3\,dt$$

$\frac{3.25}{\cancel{\text{Kfa}}\ 121}$:

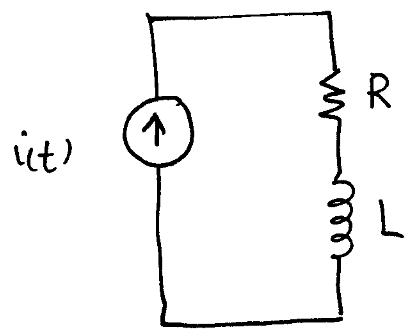


$$i_1 = 6\cos 3t$$

$$4\,\frac{d(i_2 - i_1)}{dt} + \frac{i_2 - i_3}{3} + \frac{1}{5}\int i_2\,dt = 0.$$

$$\cancel{1}\,\frac{di_3}{dt} + \frac{1}{3}(i_3 - i_2) + \frac{1}{2}(i_3 - i_1) = 0.$$

$\frac{E\ 12}{24}$:

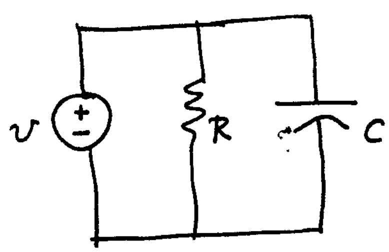


$$i(t) = I_m \sin \omega t$$

(a) $v_R = i(t)\,R = I_m R \sin \omega t$

(b) $v_L = L\frac{di}{dt} = L I_m \omega \cos \omega t$

$\frac{E\ 14}{24}$:



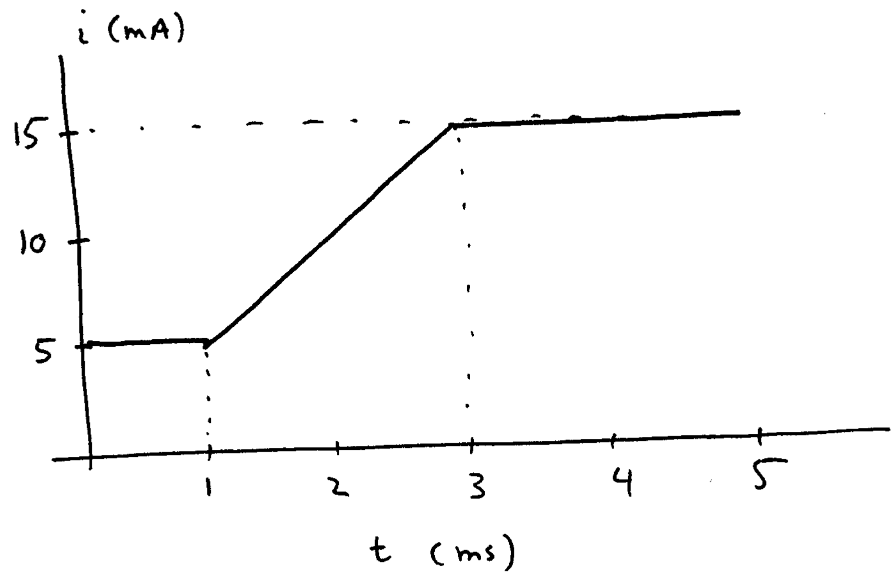
$v(t) = 2000t$ v , $R = 4\,\Omega$, $C = 300\,\mu F$

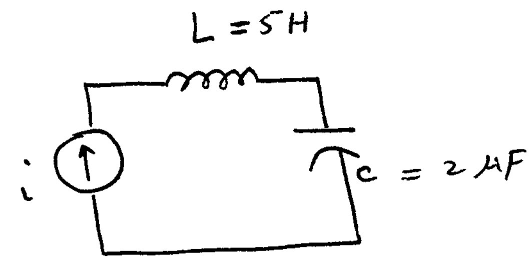
$$i_R = \frac{v}{R} = \frac{2000t}{4} = 500t\Big|_{t=5m.} = 2.5 \text{ A}$$

$$i_C = C\frac{dv}{dt} = (300\times10^{-6})(2\times10^{3}) = .6 \text{ A}$$

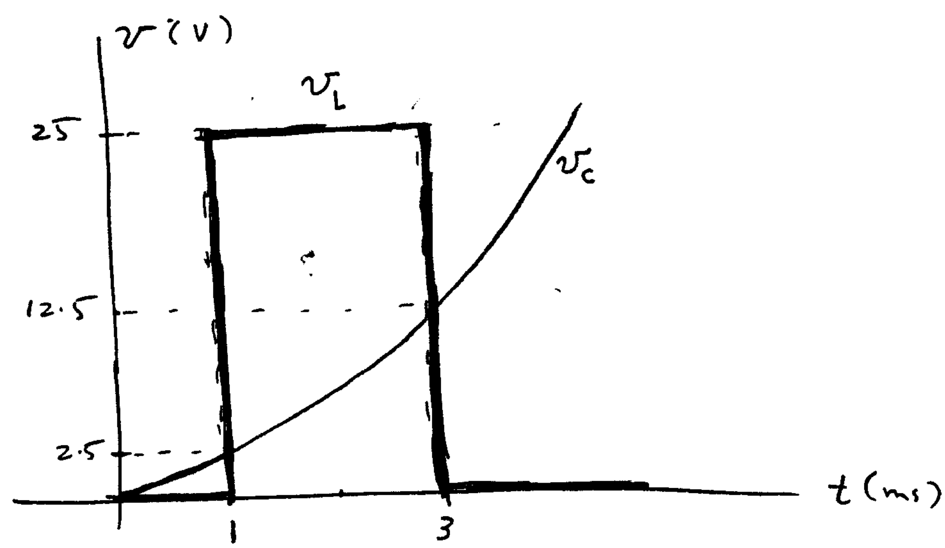
E15 :

25



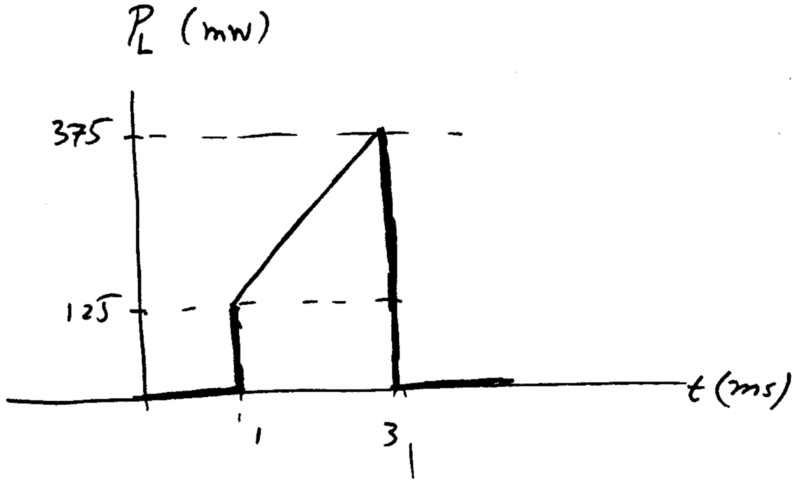


$$v_L = L\frac{di}{dt} \quad , \quad v_c = \frac{1}{c}\int i\, dt$$

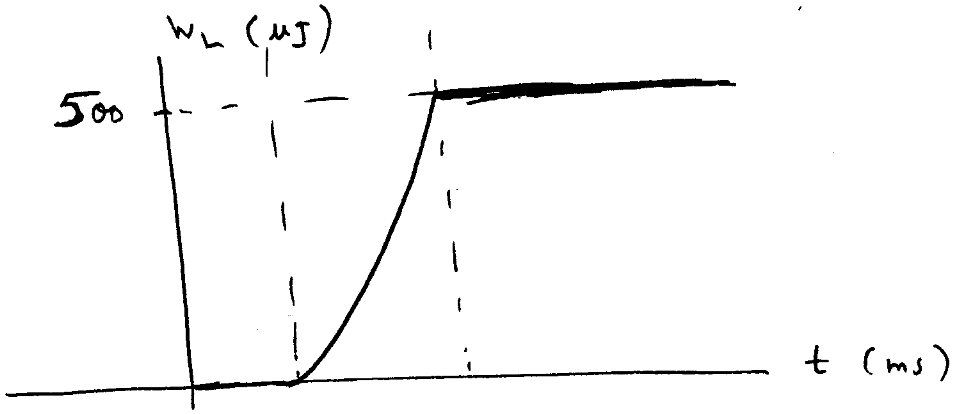


$$v_c = \frac{1}{2\times10^{-6}}\int_0^{10^{-3}} \underbrace{(5\times10^{-3})}_{A} dt = 2.5 \quad V$$

E 18 :
25



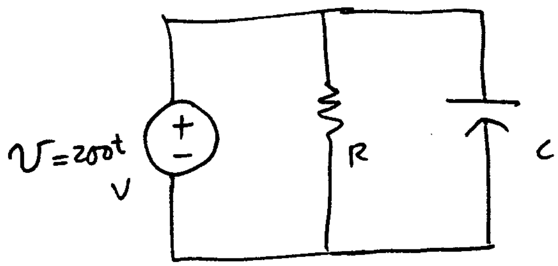
$$W_L = \int P_L \, dt$$



$\frac{E\,21}{25}$:

$t = .5\ s$, $R = 200\ \Omega$

$C = 1000\ \mu F$



(a) $$P_R = \frac{v^2}{R} = \frac{(200t)^2}{200}\Bigg|_{t=.5s} = \frac{10^4}{200} = \underline{50\ W.}$$

(b) $$P_c = i\,v = C\,\frac{dv}{dt}\,v = (1000\times10^{-6})(200)(200\times.5) = \underline{20\ W}$$

(c) $$P_s = P_R + P_c = \underline{70}\ W.$$

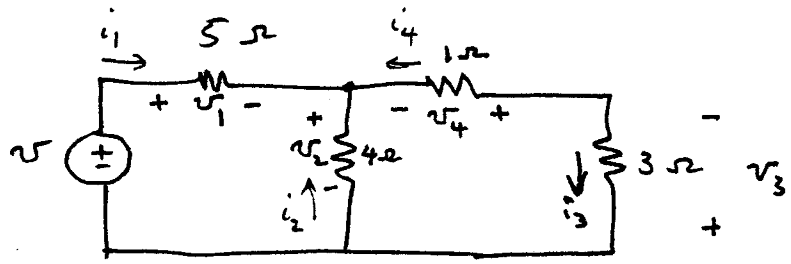
(d) $$W_R = \int_0^{.5} P_R\,dt = \int_0^{.5} \frac{v^2}{R}\,dt = \int_0^{.5} 200t^2\,dt = \underline{8.33}\ J.$$

(e) $$W_c = \int_0^{.5} P_c\,dt = \frac{1}{2}Cv^2 = \frac{1}{2}(1000\times10^{-6})(200\times.5$$

$$= \underline{5\ J.}$$

109 (Bobrow)

39



a) $i_1 = 4$ A, $v_1 = ?$

$$v_1 = 5 i_1 = 20 \text{ V.}$$

b) $i_2 = -2$ A, $v_2 = ?$

$$v_2 = -4 i_2 = -4(-2) = 8 \text{ V.}$$

c) $i_3 = 2$ A, $v_3 = ?$

$$v_3 = -3 i_3 = -3(2) = -6 \text{ V.}$$

d) $i_4 = -2$ A, $v_4 = ?$

$$v_4 = 1(i_4) = -2 \text{ V.}$$