Inductance and Capacitance

Inductors are circuit elements based on phenomena associated with magnetic fields. The source of the magnetic field is charge in motion, or current. If the current is varying with time, the magnetic field is varying with time. A time-varying magnetic field induces a voltage in any conductor that is linked by the field. The circuit parameter of inductance relates the induced voltage to the current.

Capacitors are circuit elements based on phenomena associated with electric fields. The source of the electric field is separation of charge, or voltage. If the voltage is varying with time, the electric field is varying with time. A time-varying electric field produces a displacement current in the space occupied by the field. The circuit parameter of capacitance relates the displacement current to the voltage.

Remarks
Energy can be stored in both magnetic and electric fields. Therefore, inductors and capacitors are passive elements that are capable of storing energy.
\[ V = L \frac{di}{dt} \]

\[ V \, dt = L \left( \frac{di}{dt} \right) \, dt = L \, di \]

\[ \int_{t_0}^{t} V \, d\tau = L \int_{i(t_0)}^{i(t)} dx = L \left( i(t) - i(t_0) \right) \]

\[ i(t) = \frac{1}{L} \int_{t_0}^{t} V \, d\tau + i(t_0) \]

\[ P = \frac{d}{dt} \frac{i}{L} = \frac{V}{L} \left[ \frac{1}{L} \int_{t_0}^{t} V \, d\tau + i(t_0) \right] \]
Also, \( P = v_i = L \frac{di}{dt} \).

\[ P = \frac{dw}{dt} = L i \frac{di}{dt} \]

\[ \therefore dw = L i \, di \]

\[ \int_{0}^{w} dx = L \int_{0}^{i} y \, dy \]

\[ \therefore w = \frac{1}{2} Li^2. \]
the capacitor

\[ i \Rightarrow \frac{C}{o} \quad + \quad \nu \quad - \]

\[ i = C \frac{\nu}{dt} \]

\[ \int_i \frac{\nu}{dt} = C \quad d\nu \Rightarrow \frac{1}{C} \int_i \frac{\nu}{dt} = d\nu \]

\[ \frac{1}{C} \int_{t_0}^{t} i \; d\tau = \int_{t_0}^{t} d\nu = \nu(t) - \nu(t_0) \]

\[ \therefore \nu(t) = \frac{1}{C} \int_{t_0}^{t} i \; d\tau + \nu(t_0) \]

\[ P = \nu \frac{\nu}{dt} = C \nu \frac{\nu}{dt} \]

\[ C \frac{\nu}{dt} \]
or

\[ P = i \left[ \frac{1}{c} \int_{t_0}^{t} i \, dt + v(t_0) \right]. \]

\[ P = \frac{dw}{dt} = 0 \quad dw = P \, dt \]  \hspace{1cm} (1)

Also, \[ P = CV \frac{dv}{dt} = 0 \quad P \, dt = \left( CV \, dv \right) \]  \hspace{1cm} (2)

(2) \quad \Rightarrow \quad (1) = 0

\[ \int_{0}^{w} dv = C \int_{y}^{u} y \, dy = 0 \quad W = \frac{1}{2} CV^2 \]
Series - Parallel Combination of \( L \& C \).

\[
\begin{align*}
\mathbf{u} &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_{eq} \\
\rightarrow_1 &= L_1 \frac{di}{dt} \\
\rightarrow_2 &= L_2 \frac{di}{dt} \\
\rightarrow_3 &= L_3 \frac{di}{dt} \\
\mathbf{v} &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = (L_1 + L_2 + L_3) \frac{di}{dt} \\
\frac{L_{eq}}{dt} \\
\therefore \quad L_{eq} &= L_1 + L_2 + L_3.
\end{align*}
\]
\[i = i_1 + i_2 + i_3 = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right) \int_{t_0}^{t} v \, d\tau + \left(i_1(t_0) + i_2(t_0) + i_3(t_0)\right).
\]

\[i = \frac{1}{L_{eq}} \int_{t_0}^{t} v \, d\tau + i(t_0).
\]

\[L_{eq} = \frac{1}{\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}}.
\]

\[i(t_0) = i_1(t_0) + i_2(t_0) + i_3(t_0).
\]
\[ \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \]

\[ \mathbf{v}_i = \frac{1}{c_i} \int_{t_0}^{t} i \, d\tau + \mathbf{v}_i(t_0) \quad ; \quad i = 1, 2, 3. \]

\[ \mathbf{v} = \left( \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} \right) \int_{t_0}^{t} i \, d\tau + \left( \mathbf{v}_1(t_0) + \mathbf{v}_2(t_0) + \mathbf{v}_3(t_0) \right) \]

\[ = \frac{1}{c_{eq}} \int_{t_0}^{t} i \, d\tau + \mathbf{v}_{eq}(t_0) \]

\[ \frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} \]

\[ \mathbf{v}_{eq}(t_0) = \mathbf{v}_1(t_0) + \mathbf{v}_2(t_0) + \mathbf{v}_3(t_0). \]
\[ i = i_1 + i_2 + i_3 \]

\[ = c_1 \frac{dv}{dt} + c_2 \frac{dv}{dt} + c_3 \frac{dv}{dt} \]

\[ i = c \frac{dv}{dt} \]

\[ \therefore c = c_1 + c_2 + c_3 \]
\[ i = 10 t e^{-5t} u(t). \]

\[ v = \int \frac{di}{dt} = (0.1) \frac{d}{dt} (10 t e^{-5t}) = e^{-5t} (1 - 5t) u(t). \]
\[ V(t) = 20t e^{-10t} u(t) \]

\[ i(t) = \frac{1}{L} \int_0^t V(t') \, dt' + i(0) = 2 \left( 1 - 10t e^{-10t} - e^{-10t} \right) A \quad t > 0 \]

Note: The variable \( i \) changes to \( I \) after discussing the energy.
\[ v(t) = 2e^{-t} u(t) \]
\[ i(t) = 2(1 - e^{-t} - e^{-2t}) u(t) \]
\[ p(t) = v(t) i(t) \]
\[ w(t) = \int p(t) \, dt \]

Note that as \( t \to \infty \), the energy decays.

Because \( i(t) \) is defined as the source of energy, the stored energy cannot dissipate after \( v \to 0 \). Therefore, a sustained current circulates in the circuit.
\[ i = 10 t e^{-5t} u(t) \]
\[ v = e^{-5t} (1 - 5t) u(t) \]
\[ p = v_i \]
\[ w = \int p \, dt \]

\textbf{Figure 6.8} The variables \( i, v, p, \) and \( w \) versus \( t \) for Example 6.1.
\[ i(t) = C \frac{dV}{dt} = (0.5 \mu F) \frac{dV}{dt} \]
\[ P = \nu i \]

- Energy is being stored in C whenever \( P > 0 \); i.e., \( 0 < t < 1 \).
- Energy is being delivered by C whenever \( P < 0 \); i.e., \( 1 < t \).

This is also seen from the graph of \( W \) below.

\[ W = \frac{1}{2} C \nu^2 \]

\[ W(\nu S) \]

Note: \[ \int_{0}^{t} P dt = W \]
\[ V(t) = 0 \quad t \leq 0 \]

\[ V(t) = \frac{1}{C} \int_0^t i(\tau) \, d\tau + V(0) \quad 0 \leq t \leq 20 \, \mu s \]

\[ = \frac{1}{(0.2)(10^{-6})} \int_0^t (5000 \, t) \, d\tau + 0 = 12.5 \times 10^9 \, t^2 \, V \]

\[ V(t) = \frac{1}{C} \int_{20 \, \mu s}^t i(\tau) \, d\tau + V(20 \, \mu s) \quad 20 \leq t \leq 40 \, \mu s \]

\[ = \frac{1}{(0.2)(10^{-6})} \int_{20 \, \mu s}^t (0.2 \cdot 5000 \, t) \, d\tau + 5 = (10^6 + 12.5 \times 10^9 \, t^2 - 10) \, V \]
\[ U(t) = \frac{1}{c} \int_{40\mu s}^{t} i(\tau) d\tau + U(40\mu s), \quad t \geq 40\mu s \]

Note that \( P > 0 \) for all \( t \).

This implies that energy is stored in the capacitor continuously, even when \( i(\tau) \) returns to zero. The energy stored in the capacitor is:

\[ W = \frac{1}{2} CV^2 \]

The voltage \( V \) remains on the capacitor after \( i(\tau) \) returns to zero.
the inductor: (Review)

\[ V = L \frac{di}{dt} \]

\[ W_L(t) = \frac{1}{2} L i(t)^2 \]

\[ W_L(t) = \int V(t) i(t) \, dt \]

\[ = \int L \frac{di}{dt} i(t) \, dt = \frac{1}{2} L i(t)^2 \]

\[ V = L \frac{d}{dt} \frac{di}{dt} = z \frac{di}{dt} \]

\[ W_L = \frac{1}{2} L i(t)^2 = i(t) \]

\[ P(t) = \frac{R}{R} \frac{i}{i} = i^2 R_2 \]

\[ V_R(t) = i R = z i \]

\[ -V_s + V_R + V = 0 \]

\[ V_s = V_R + V \]

\[ V_s = \frac{V_R}{2} + V \]

---

# 3.1

\[ \frac{2}{15} \]

\[ V(s) \]

\[ \frac{2}{R} \]

\[ i(t) \]

\[ \frac{1}{R} \]

\[ \frac{2}{R} \]

\[ \frac{2}{2} \]

\[ \frac{2}{2} \]

\[ \frac{2}{2} \]

\[ \frac{2}{2} \]

\[ \frac{2}{2} \]

\[ t \]

\[ t \]
\[ V_R = iR = i \]
\[ V_L = L \frac{di}{dt} = \frac{1}{3} \frac{di}{dt} \]
\[ V = V_R + V_L \]
The capacitor (review)

\[ i = \frac{C}{v} \]

\[ i = C \frac{dv}{dt} \]

\[ W_c(t) = \frac{1}{2} C v^2(t) \]

**Differentiator:**

\[ C \frac{dv_c}{dt} = i = \frac{v_c - v_0}{R} \]

\[ v_c = v_i \]

\[ v_0 = -RC \frac{dv_i}{dt} \]
Integrator (3.22/16)

\[ V_i \frac{dV_o}{dt} = 0 \quad V_o = -\frac{1}{RC} \int v_i \, dt \]

**Important Remark:**

Consider an inductor

\[ i(t) = \frac{1}{L} \int_{-\infty}^{t} v(t) \, dt \]

Then at \( t = a \),

\[ i(a) = \frac{1}{L} \int_{-\infty}^{a} v(t) \, dt \]

\[ i(a+\epsilon) = \frac{1}{L} \int_{-\infty}^{a+\epsilon} v(t) \, dt = \frac{1}{L} \int_{-\infty}^{a} v(t) \, dt + \frac{1}{L} \int_{a}^{a+\epsilon} v(t) \, dt \]

\[ = i(a) + \frac{1}{L} \int_{a}^{\epsilon} v(t) \, dt \]

If \( \epsilon \to 0 \)

\[ i(a+\epsilon) = i(a) \]
The current through an inductor cannot change instantaneously. This fact can be described as follows:

\[ W_L = \frac{1}{2} L i(t)^2 \quad \text{and} \quad P_L = \frac{d}{dt} W_L \]

Note that a rapid change in \( i(t) \) requires an infinite power which is not physically realizable.

In a similar manner, it can be shown that the voltage across a capacitor cannot change instantaneously.
\[ i = \frac{1}{L} \int v(t) \, dt \]
Important Circuit Concepts

Just as we analyzed circuits with R's and energy sources, so can we write a set of node or mesh equations for circuits that contain L's and C's.

Example:

3.23:

\[ 6 \cos t \]

\[ 3 \Omega \]

\[ 2 \Omega \]

\[ 4F \]

\[ 5 \Omega \]

\[ \begin{align*}
6 \cos t &= \frac{v_1}{3} + \int (v_1 - v_2) \, dt \\
\int (v_1 - v_2) \, dt &= \frac{v_2}{2} - v_3 \\
\frac{v_2 - v_3}{2} &= \int \frac{dv_2}{dt} + \frac{v_3}{5}
\end{align*} \]
\[ \frac{7 \sin 4t - V_1}{2} = \int (V_1 - V_2) \, dt \]

\[ \int (V_1 - V_2) \, dt + 3 \frac{d}{dt} \left( \frac{7 \sin 4t - V_2}{2} \right) = 4 \frac{dV_3}{dt} + \frac{V_2 - V_3}{5} \]

\[ \frac{V_2 - V_3}{5} = \frac{1}{6} \int v_3 \, dt \]

\[ i_1 = 6Cn^3t \]

\[ 4 \frac{d(i_2 - i_1)}{dt} + \frac{i_2 - i_3}{3} + \frac{1}{5} \int i_2 \, dt = 0. \]

\[ \frac{d}{dt} \left( i_3 - i_2 \right) + \frac{1}{3} (i_3 - i_2) + \frac{1}{2} (i_3 - i_1) = 0. \]
\( \frac{E}{2Y} \)

\[ i(t) = I_m \sin \omega t \]

(a) \[ V = R \frac{di}{dt} = I_m R \sin \omega t \]

(b) \[ V_L = L \frac{di}{dt} = L I_m \omega C \sin \omega t \]

\( E_{14} \)

\[ V(t) = 2000t \quad \text{V} \quad \quad R = 4 \quad \Omega \quad \quad C = 300 \quad \mu F \]

\[ i_R = \frac{V}{R} = \frac{2000t}{4} = 500t \quad \left| \begin{array}{c}
\text{t=5m}.
\end{array}\right. = 2.5 \quad \text{A} \]

\[ i_C = C \frac{dv}{dt} = (300 \times 10^{-6}) (2 \times 10^3) = 0.6 \quad \text{A} \]
\[ E_{15} \]

\[
\begin{align*}
\mathbf{L} &= 5\, \text{H} \\
\mathbf{c} &= 2\, \text{\mu F}
\end{align*}
\]

\[
\begin{align*}
\mathbf{v}_L &= L \frac{\mathbf{d} \mathbf{i}}{\mathbf{d} \mathbf{t}} \\
\mathbf{v}_c &= \frac{1}{\mathbf{c}} \int \mathbf{i} \, d\mathbf{t}
\end{align*}
\]

\[
\mathbf{v}_c = \frac{1}{2 \times 10^{-6}} \int_{0}^{10^{-3}} (5 \times 10^2) \, d\mathbf{t} = 2.5 \, \text{V}
\]
\[ E \frac{18}{25} \]

\[ P_L (\text{mW}) \]

\[ W_L = \int P_L \, dt \]

\[ W_L (\mu J) \]

\[ 500 \]

\[ t (\text{ms}) \]
\[
\begin{align*}
\text{(a)} \quad P_R &= \frac{V^2}{R} = \frac{(200t)^2}{R} \quad \text{at} \quad t = 0.5 \text{ s}, \quad R = 200 \Omega, \quad C = 1000 \mu\text{F} \\
&= \frac{10^4}{900} = 50 \text{ W.}
\end{align*}
\]

\[
\begin{align*}
\text{(b)} \quad P_c &= i \cdot V = C \frac{dV}{dt} \quad V = (1000 \times 10^{-6})(200)(200 \times 0.5) = 20 \text{ V}
\end{align*}
\]

\[
\begin{align*}
\text{(c)} \quad P_s &= P_R + P_c = 70 \text{ W.}
\end{align*}
\]

\[
\begin{align*}
\text{(d)} \quad W_R &= \int_0^{0.5} P_R \, dt = \int_0^{0.5} \frac{V^2}{R} \, dt = \int_0^{0.5} 200t^2 \, dt = 8.33 \text{ J.}
\end{align*}
\]

\[
\begin{align*}
\text{(e)} \quad W_C &= \int_0^{0.5} P_c \, dt = \frac{1}{2} C V^2 = \frac{1}{2} (1000 \times 10^{-6})(20 \times 0.5) \\
&= 5 \text{ J.}
\end{align*}
\]
a) \( i_1 = 4 \) A, \( v_1 = ? \)

\[ v_1 = 5 \times 4 = 20 \text{ V}. \]

b) \( i_2 = -2 \) A, \( v_2 = ? \)

\[ v_2 = -4 \times (-2) = 8 \text{ V}. \]

c) \( i_3 = 2 \) A, \( v_3 = ? \)

\[ v_3 = -3 \times 2 = -6 \text{ V}. \]

d) \( i_4 = -2 \) A, \( v_4 = ? \)

\[ v_4 = 1 \times (-2) = -2 \text{ V}. \]