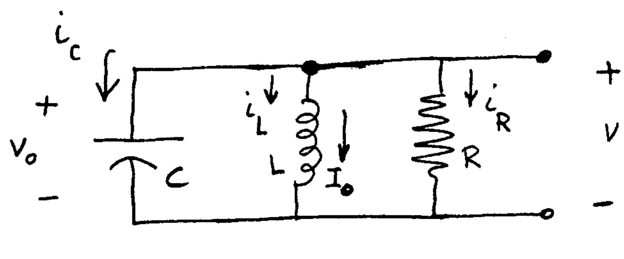
Chapter 8.

RLC Circuits

1. Natural response of parallel RLC.



$$i_R + i_L + i_c = 0$$

$$\underbrace{\frac{v}{R}}_{i_R} + \underbrace{\frac{1}{L}\int_0^t v(\tau)\,d\tau + I_0}_{i_L} + \underbrace{C\frac{dv}{dt}}_{i_c} = 0$$

$$\frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0.$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0.$$

$$v = Ae^{st}, \quad \frac{dv}{dt} = Ase^{st}, \quad \frac{d^2v}{dt^2} = As^2e^{st}$$

$$As^2e^{st} + \frac{1}{RC}Ase^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\underbrace{\left[s^2 + \frac{s}{RC} + \frac{1}{LC}\right]}_{\text{characteristic eq}^n} = 0.$$

$$S_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

α $\quad$ ω_0^2

$$v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$$

$$\alpha = \frac{1}{2RC} \quad , \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$S_{1,2}$: Complex Freq.

α : Neper Freq.

ω_0 : resonant radian Freq.

there are three cases :

1. $\alpha^2 - \omega_0^2 > 0$ ← real, distinct roots $\Longrightarrow$ $\omega_0 < \alpha$ (overdamped)

2. $\alpha^2 - \omega_0^2 < 0$ ← complex conjugate roots $\Longrightarrow$ $\omega_0 > \alpha$ (Underdamped)

3. $\alpha^2 - \omega_0^2 = 0$ ← two equal roots $\Longrightarrow$ $\omega_0 = \alpha$ (Critically damped)

8.1

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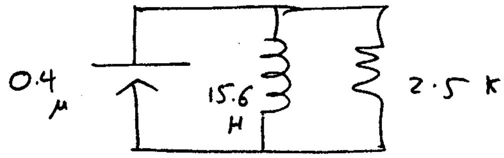


(a) $$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0$$

$$s^2 + \frac{1}{\cancel{(2.5)}(0.4)(10^3)(10^{-6})} s + \frac{1}{(15.6)(0.4)(10^{-6})} = 0$$

$$s^2 + 1000 s + 160'000 = 0$$

$$s_{1,2} = -500 \pm \sqrt{(500)^2 - 160000} \quad \begin{matrix} \nearrow -200 \\ \searrow -800 \end{matrix}$$

(b) Overdamp

(c) $$\sqrt{-\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} = 500 \Rightarrow L = 5\ H$$

(d) $$\left(\frac{1}{2RC}\right)^2 = \frac{1}{LC} \Rightarrow R = 3125\ \Omega$$

1. <u>Overdamped Voltage Response</u> : (s_1, s_2 are real - distinct)

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0$$

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v(0) = v_0 \Rightarrow \quad A_1 + A_2 = v_0 \qquad (1)$$

also in

$$\frac{V}{R} + \frac{1}{L}\int_0^t v(\tau)\,d\tau + I_0 + C\frac{dv}{dt} = 0 \qquad t \geq 0$$

at $t=0$ we have

<u>initial Conditions</u>

$$\frac{\cancel{v(0)}\, V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0.$$

$$\frac{dv(0)}{dt} = -\frac{V_0}{RC} - \frac{I_0}{C}$$

$$\left.\frac{dv}{dt}\right|_{t=0} = \left. A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}\right|_{t=0} = A_1 s_1 + A_2 s_2$$

$$A_1 s_1 + A_2 s_2 = -\frac{V_0}{RC} - \frac{I_0}{C} \qquad (2)$$

from (1) and (2)

$$\Rightarrow \quad A_{1,2} = \pm \frac{V_0 S_{2,1} + \left(\frac{1}{C}\right)\left[\frac{V_0}{R} + I_0\right]}{S_2 - S_1}$$

2. Under damped Voltage Response (s_1, s_2 Complex conjugate)

$$s_{1,2} = -\alpha \pm j\underbrace{\sqrt{\omega_0^2-\alpha^2}}_{\omega_d}$$

ω_d : damped radian freq.

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$= A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t}$$

$$= e^{-\alpha t}\left[A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right]$$

$$= e^{-\alpha t}\left[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)\right]$$

$$= e^{-\alpha t}\left[\underbrace{(A_1 + A_2)}_{B_1}\cos\omega_d t + \underbrace{(A_1 j - A_2 j)}_{B_2}\sin\omega_d t\right]$$

real part

imaginary part

$$= e^{-\alpha t}\left[B_1 \cos\omega_d t + B_2 \sin\omega_d t\right]$$

But
$$\begin{cases} v(0) = V_0 \\ \dfrac{dv(0)}{dt} = -\dfrac{V_0}{RC} - \dfrac{I_0}{C} \end{cases}$$

$$\therefore \quad V_0 = B_1$$

$$\left.\frac{dv}{dt}\right|_{t=0} = \omega_d B_2 - \alpha \cancel{B_1}^{V_0} = -\frac{V_0}{RC} - \frac{I_0}{C}$$

$$\therefore \quad B_2 = -\frac{\alpha}{\omega_d}\left(V_0 + 2 I_0 R\right).$$

$$\therefore \quad v(t) = V_0 e^{-\alpha t} \cos \omega_d t - \frac{\alpha}{\omega_d}\left(V_0 + 2 I_0 R\right) e^{-\alpha t} \sin \omega_d t$$

3. Critically Damped Voltage Response:

$(s_1 = s_2)$

$\therefore \quad s_1 = s_2 = -\alpha = -\dfrac{1}{2RC}$

$v(t) = D_2 e^{s_1 t} + D_1 t e^{s_2 t}$

$= D_2 e^{-\alpha t} + D_1 t e^{-\alpha t}$

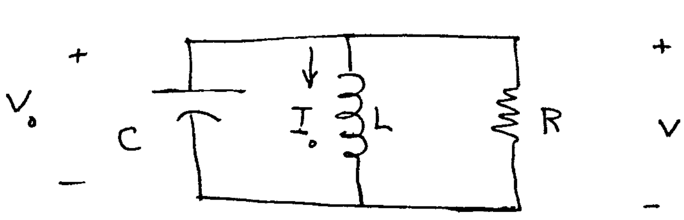
$v(0) = V_0 = D_2$

$v'(0) = -\dfrac{V_0}{RC} - \dfrac{I_0}{C} = D_1 - \alpha D_2$

$D_1 = -\alpha (V_0 + 2 I_0 R)$

$\alpha = \dfrac{1}{2RC}$

Review :



$$\left.\begin{aligned}&\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC} v = 0.\\ &v(0) = V_0 \quad , \quad v'(0) = -\frac{1}{C}\left(\frac{V_0}{R} + I_0\right)\end{aligned}\right\} \leftarrow \text{I.V.P}$$

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0.$$

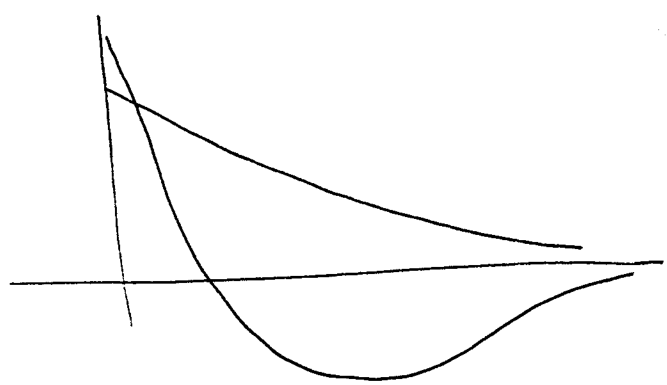
ⓐ $\underline{s_{1,2} \quad \text{real, distinct}}$ $\qquad$ (overdamped)

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where

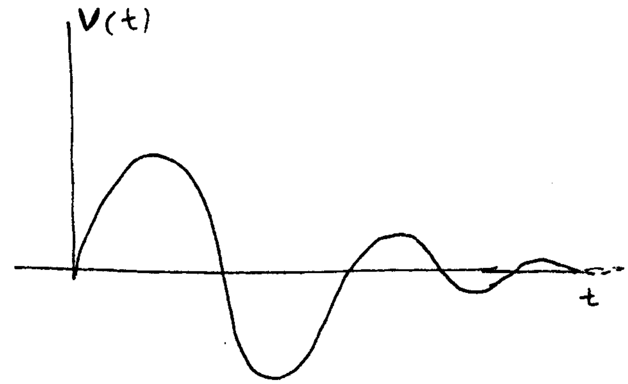
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\begin{cases} \alpha = \dfrac{1}{2RC} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$$



(b) $s_{1,2}$ complex conjugate (underdamped)

$$V(t) = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)$$



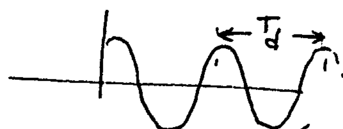
where

$$s_{1,2} = -\alpha \pm j\,\omega_d$$

$$\begin{cases} \alpha = \dfrac{1}{2RC} \\ \omega_d = \sqrt{\omega_0^2 - \alpha^2} \end{cases}$$

* note: If $\alpha \to 0$ $(R \to \infty)$ $\overset{\omega_d = \omega_0}{\Rightarrow}$ $V(t) = B_1 \cos\omega_0 t + B_2 \sin\omega_0 t$

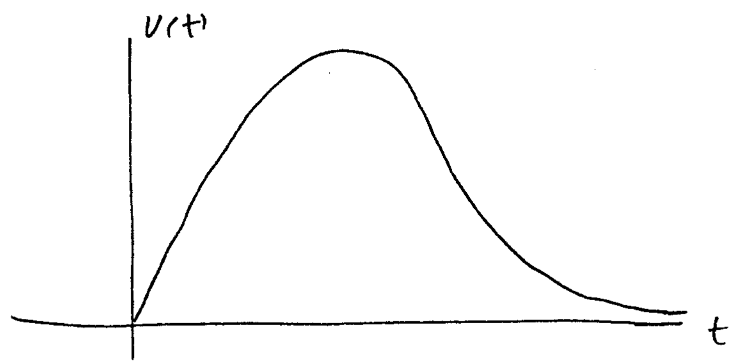
$f_d = \dfrac{1}{T_d}$



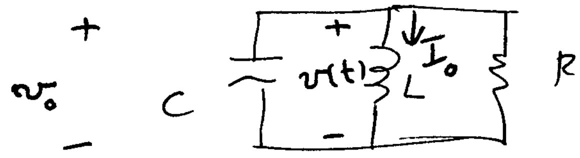
(c) $s_1 = s_2$ repeated roots (Critically Damped)

$$V(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

$$\alpha = \frac{1}{2RC}$$



$\#\frac{8.2}{268}$:



$$\begin{cases} V_o = 0 \\ I_o = 18\ mA \end{cases}$$

$$v(t) = 15e^{-8000t} - 15e^{-2000t}$$

$$R, L, C = ?$$

$$C\frac{dv}{dt} + \frac{1}{L}\int_0^t v(\tau)\, d\tau + I_o + \frac{V}{R}\bigg|_{t=0} = 0$$

$$\therefore \quad C\frac{d v(0)}{dt} + \overset{.018}{\cancel{I_o}} + \overset{0}{\cancel{\frac{V(0)}{R}}} = 0 \qquad (1)$$

$$v(t) = 15e^{\overset{s_1}{-8000}t} - 15e^{\overset{s_2}{-2000}t}$$

$$\frac{dv}{dt}\bigg|_{t=0} = (15)(-8000) + (15)(2000) \qquad (2)$$

$$(1),\ (2) \Rightarrow \underline{\underline{C = 0.2}}\ \mu F.$$

$$\begin{cases} s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \\ s_1 + s_2 = -2\alpha \\ |s_1 - s_2| = 2\sqrt{\alpha^2 - \omega_0^2} \end{cases}$$

$$s_1 + s_2 = -2\alpha = -8000 - 2000 \Rightarrow \alpha = 5000$$

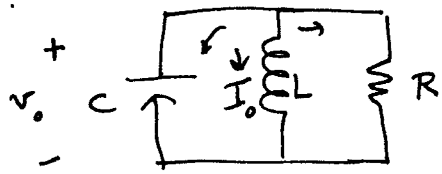
$$\alpha = \frac{1}{2RC} \Rightarrow \underline{\underline{R = 500\ \Omega}}$$

$$|s_1 - s_2| = 2\sqrt{\alpha^2 - \omega_0^2} = 6000 \Rightarrow \omega_0 = 4000\ \frac{rad}{s}$$

$$\omega_0^2 = \frac{1}{LC} \Rightarrow \underline{\underline{L = 312.5\ mH}}$$

//

$\frac{8.3}{268}$:



$R = 2k$

$C = .25\ \mu F$

$L = 6.25\ H.$

$I_o = 0.$

$v_o = -60\ V$

a) $i_L(0^+) = 0$

$i_R(0^+) = \frac{-60}{2k} = -30\ mA$

$i_C(0^+) = -\left(i_R(0^+) + \cancel{i_L(0^+)}\right) = 30\ mA$

b) $\alpha = \frac{1}{2RC} = \frac{1}{2(2000)(.25)(10^{-6})} = 1000.$

$\omega_o^2 = \frac{1}{LC} = \frac{1}{(6.25)(.25)(10^{-6})} = 64 \times 10^4$

$\omega_o = 800.\ \frac{rad}{sec}.$

$\alpha^2 > \omega_o^2 \quad \therefore \quad$ over ~~damped~~.

$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$ → $-400.$ / → $-1600.$

$v(t) = A_1 e^{-400t} + A_2 e^{-1600t}. \qquad t \geq 0$

$$v(0) = A_1 + A_2 = -60$$

$$i_c(0^+) = C\frac{dv(0^+)}{dt} = \underbrace{\cancel{C}}_{.25\times10^{-6}}\left[-400A_1 - 1600A_2\right] = 30\times10^{-3}$$

$\therefore$ $A_1 = 20$ V

$A_2 = -80$ V

$\therefore$ $v(t) = 20e^{-400t} - 80e^{-1600t}$; $t \geq 0$.

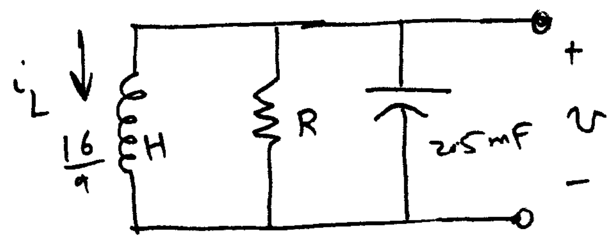
c) $i_L(t) = -(i_R + i_c)$

$$i_R = \frac{v(t)}{R} = 10e^{-400t} - 40e^{-1600t} \quad \text{mA}$$

$$i_c = C\frac{dv}{dt} = -2e^{-400t} + 32e^{-1600t} \quad \text{mA}$$

$$i_L(t) = -8e^{-400t} + 8e^{-1600t}, \quad \text{mA} ; \quad t \geq 0^+$$

$\# \frac{8.4}{269}$:



$R = ?$

$i_R(0^+)$, $i_L(0^+)$, $i_C(0^+) = ?$

$$v(t) = 50e^{-5t} - 90e^{-45t} \quad V \quad ; \quad t \geq 0$$

$$i_L(0^+) = ?$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - w_0^2} \quad \begin{matrix} \to -5 \\ \to -45 \end{matrix}$$

$$s_1 + s_2 = -2\alpha = -50$$

$$\alpha = 25 = \frac{1}{2RC} \implies R = 8\,\Omega$$

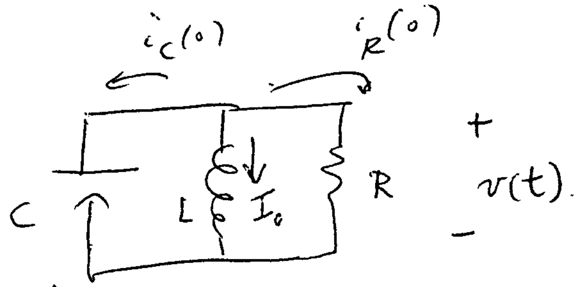
$$v(0^+) = 50 - 90 = -40 \quad A.$$

$$i_R(0^+) = \frac{-40}{8} = \underline{-5 \ A}$$

$$i_C(0^+) = C\,\frac{d\,v(0)}{dt} = 2.5\times10^{-3}(-250 + 4050) = 9.5A$$

$$i_L(0^+) = -9.5 + 5 = \underline{-4.5 \ A}$$

8.5 / 269 :



$$v = 10e^{\overbrace{-1000}^{-\alpha}t}(4\cos \underbrace{5000}_{\omega_d}t - \sin 5000t)$$

$C = 0.5\ \mu F$

(a) $R = ?$

$$\alpha = 1000 = \frac{1}{2RC} \Rightarrow R = \frac{10^6}{(2000)(0.5)}$$

$$= 1\ K\Omega$$

(b) $\omega_D = \sqrt{\omega_0^2 - \alpha^2} = 5000$

$$\omega_0^2 = 25 \times 10^6 + 10^6 = 26 \times 10^6 = \frac{1}{LC}$$

$$L = \frac{1}{(26)(\cancel{10^6})(0.5)(\cancel{10^{-6}})} = \frac{1}{13} = 0.0769\ H$$

(c) $V_0 = v(0) = 40\ V$

(d) $I_0 = -(i_c(0) + i_R(0))$

$$i_R(0) = \frac{V_0}{R} = \frac{40}{1} = 40 \quad mA$$

$$i_C(0) = i_C(t)\Big|_{t=0} = C\,\frac{dv}{dt}\Big|_{t=0} = 6.5)\left[\;- - \quad - -\;\right]\Big|_{t=0} = -45 \; mA$$

$$\therefore \quad I_0 = -(-45+40) = 5 \quad mA.$$

(e) $$i_L(t) = \frac{1}{L}\int_0^t v(\tau)\,d\tau + I_0$$

or

$$i_L(t) = -\left(i_C(t) + i_R(t)\right)$$

$\# \frac{8.6}{269}$:

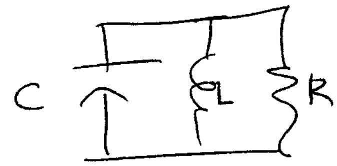
$R = 5K$

$L = 8\,H$

$C = .125\ \mu F$

$V_0 = 30\,V$

$I_0 = 6\,mA$



(a) $s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0$

$$s^2 + \frac{1}{(5 \times 10^3)(.125)(10^{-6})} s + \frac{1}{(8)(.125)(10^{-6})} = 0$$

$$s^2 + 1600 s + 1000000 = 0$$

$$s_{1,2} = -800 \pm \sqrt{(800)^2 - 1000000} = \underbrace{-800 \pm j\,600}_{-\alpha \pm j\omega_d}$$

$$v(t) = e^{-800t}\left[B_1 \cos 600t + B_2 \sin 600t \right]$$

$$B_1 = v(0) = \underline{\underline{30\,v}}$$

$$v'(0) = -\frac{1}{C}\left(\frac{V_0}{R} + I_0 \right) = \omega_d B_2 - \alpha B_1 \Rightarrow \underline{\underline{B_2 = -120}}$$

$$v(t) = 30 e^{-800t}\left(\cos 600t - 4 \sin 600t \right) \quad ; \quad t \geq 0$$

(b) $\frac{dv}{dt} = 0 \Rightarrow \tan 600t = \frac{32}{26}$

$\therefore \quad 600t = \tan^{-1} \frac{32}{26} = (50.91^\circ + k\pi)^\circ$

$k = 0, \pm 1, \pm 2, \ldots$

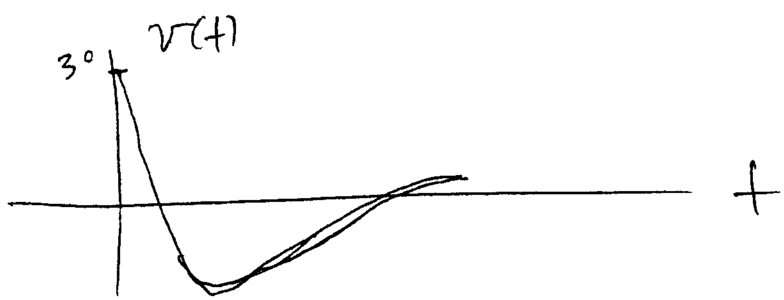
$= .8885 + k\pi$ radians , $k = 0, 1, 2, \ldots$

$t_1 = \frac{.8885}{600} = 1480.8 \ \mu s$

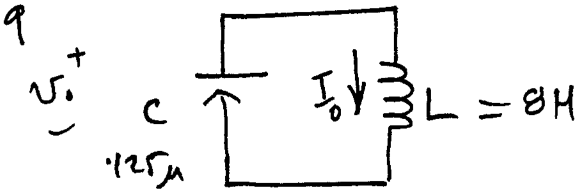
$t_2 = 6716.79 \ \mu s$

$t_3 = 11952.78 \ \mu s$

(c) $\omega_d = 600 = \frac{2\pi}{T_D} \rightarrow T_d = \frac{2\pi}{600} = t_3 - t_1$



8.7 / 269 :



a) $v_0 = 30\,V$ $\quad \omega_0 = 1000$

$I_0 = 6\,mA$

$R=\infty\,,\ \alpha=0 \Rightarrow s_{1,2} = \pm j\omega_0 = \pm j1000.$

$v(t) = B_1 \cos 1000t + B_2 \sin 1000t$

$v(0^+) = B_1 = \underline{30\ V}$ $\qquad C\dfrac{dv(0)}{dt} = -I_0 \Rightarrow \dfrac{dv(0)}{dt} = -\dfrac{I_0}{C} = -\dfrac{6\times10^{-3}}{.125\times10^{-6}}$

$\dfrac{dv(0^+)}{dt} = -48\,000 = 1000 B_2 \Rightarrow \underline{B_2 = -48\,V}$

$v(t) = 30\cos 1000t - 48\sin 1000t \quad ;\ t \geq 0$

b) $2\pi f = 1000 \quad \therefore \quad f = 159.15\ Hz.$

c) $\dfrac{dv}{dt} = -30000 \sin 1000t - 48\,000 \cos 1000t = 0$

$\tan 1000t = -1.6 \Rightarrow 1000t = 122. + 180k$

$|V_{max}| = |30\cos 122 - 48\sin 122| = \underline{56.6\ V}$