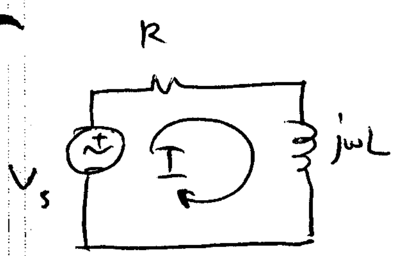
9-5 Response as a function of ω

We will now consider methods of obtaining and presenting the response of a circuit with sinusoidal excitation as a function of the radian frequency ω. With the possible exception of the 60-Hz power area in which frequency is a constant and the load is the variable, sinusoidal frequency response is extremely important in almost every branch of electrical engineering as well as in related areas, such as the theory of mechanical vibrations or automatic control.

Let us suppose that we have a circuit which is excited by a single source $\mathbf{V}_s = V_s\underline{/\theta}$. This phasor voltage may also be transformed into the time-domain source voltage $V_s \cos(\omega t + \theta)$. Somewhere in the circuit exists the desired response, say, a current $\mathbf{I}$. As we know, this phasor response is a complex number, and its value cannot be specified in general without the use of two quantities: either a real part and an imaginary part, or an amplitude and a phase angle. The latter pair of quantities is more useful and more easily determined experimentally, and is the information which we shall obtain analytically as a function of frequency. The data may be presented as two curves, the magnitude of the response as a function of ω and the phase angle of the response as a function of ω. We often normalize the curves by plotting the magnitude of the current-voltage ratio and the phase angle of the current-voltage ratio versus ω. It is evident that an alternative description of the resultant curves is the magnitude and phase angle of an admittance as a function of frequency. The admittance might be an input admittance or, if the current and voltage are measured at different locations in a circuit, a *transfer* admittance. A normalized voltage response to a current source may be similarly presented as the magnitude and phase angle of an input or transfer impedance versus ω. Other possibilities are voltage-voltage ratios (voltage gains) or current-current ratios (current gains). Let us consider the details of this process by thoroughly discussing several examples.



For the first example, we select the series RL circuit. The phasor voltage $\mathbf{V}_s$ is therefore applied to this simple circuit, and the phasor current $\mathbf{I}$ (leaving the positively marked end of the voltage source) is selected as the desired response. We are dealing with the forced response only, and the familiar phasor methods enable the current to be obtained:

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + j\omega L}$$

Let us immediately express this result in normalized form as a ratio of current to voltage, that is, as an input admittance:

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}_s}$$

or

$$\mathbf{Y} = \frac{1}{R + j\omega L} \tag{3}$$

If we like, we may consider the admittance as the current produced by a source voltage $1\underline{/0^\circ}$ V. The magnitude of the response is

$$|\mathbf{Y}| = \frac{1}{\sqrt{R^2 + \omega^2 L^2}} \tag{4}$$

while the angle of the response is found to be

$$\text{ang}\,\mathbf{Y} = -\tan^{-1}\frac{\omega L}{R} \tag{5}$$

Equations (4) and (5) are the analytical expressions for the magnitude and phase angle of the response as functions of ω; we now desire to present this same information graphically.

First consider the magnitude curve. It is important to note that we are plotting the absolute value of some quantity versus ω, and the entire curve must therefore lie *above* the ω axis. The response curve is constructed by noting that the value of the response at zero frequency is $1/R$, the initial slope is zero, and the response approaches zero as frequency approaches infinity; the graph of the magnitude of the response as a function of ω is shown in Fig. 9-13a.

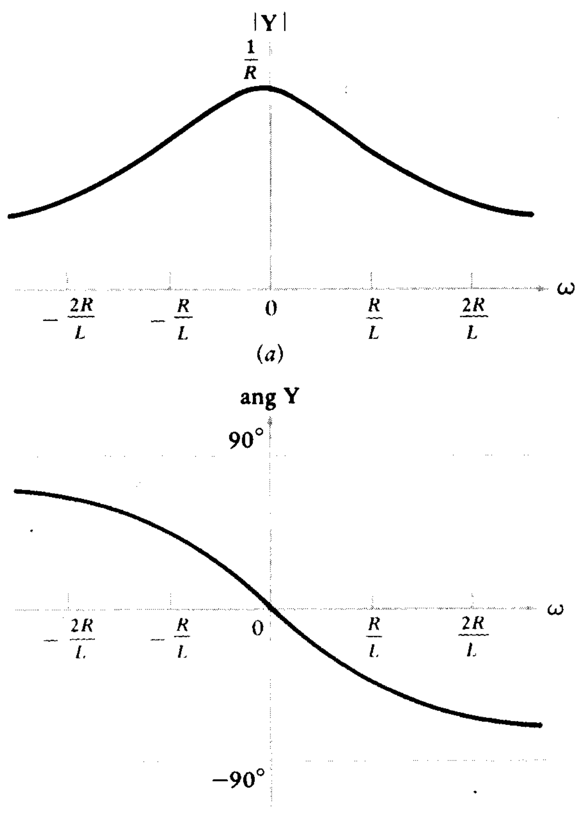


Figure 9-13

(a) The magnitude of $\mathbf{Y} = \mathbf{I}/\mathbf{V}_s$ and (b) the angle of $\mathbf{Y}$ are plotted as functions of ω for a series RL circuit with sinusoidal excitation.

For the sake of generality and completeness, the response is shown for both positive and negative values of frequency; the symmetry results from the fact that Eq. (4) indicates that $|\mathbf{Y}|$ is unchanged when ω is replaced by $(-\omega)$. The physical interpretation of a negative radian frequency, such as $\omega = -100$ rad/s, depends on the time-domain function, and it may always be obtained by inspection of the time-domain expression. Suppose, for example, that we consider the voltage $v(t) = 50 \cos(\omega t + 30°)$. If $\omega = 100$, the voltage is $v(t) = 50 \cos(100t + 30°)$, but if $\omega = -100$, $v(t) = 50 \cos(-100t + 30°)$ or $50 \cos(100t - 30°)$. These voltages have different values at $t = 1$ ms, for example. Any sinusoidal response may be treated in a similar manner.

The second part of the response, the phase angle of $\mathbf{Y}$ versus ω, is an inverse tangent function. The tangent function itself is quite familiar, and we should have no difficulty turning that curve on its side. Drawing in asymptotes of +90° and −90° is helpful. The response curve is shown in Fig. 9-13b. The points at which $\omega = \pm R/L$ are marked on both the magnitude and phase curves. At these frequencies the magnitude is 0.707 times the maximum magnitude at zero frequency and the phase angle has a magnitude of 45°. At the frequency at which the admittance magnitude is 0.707 times its maximum value, the current magnitude is 0.707 times its maximum value, and the average power supplied by the source is 0.707^2, or 0.5, times its maximum value. It is not very strange that $\omega = R/L$ is identified as a *half-power frequency.*

As a second example, let us select a parallel LC circuit driven by a sinusoidal current source, as illustrated in Fig. 9-14*a*. The voltage response $\mathbf{V}$ is easily obtained:

$$\mathbf{V} = \mathbf{I}_s \frac{(j\omega L)(1/j\omega C)}{j\omega L - j(1/\omega C)}$$

and it may be expressed as an input impedance

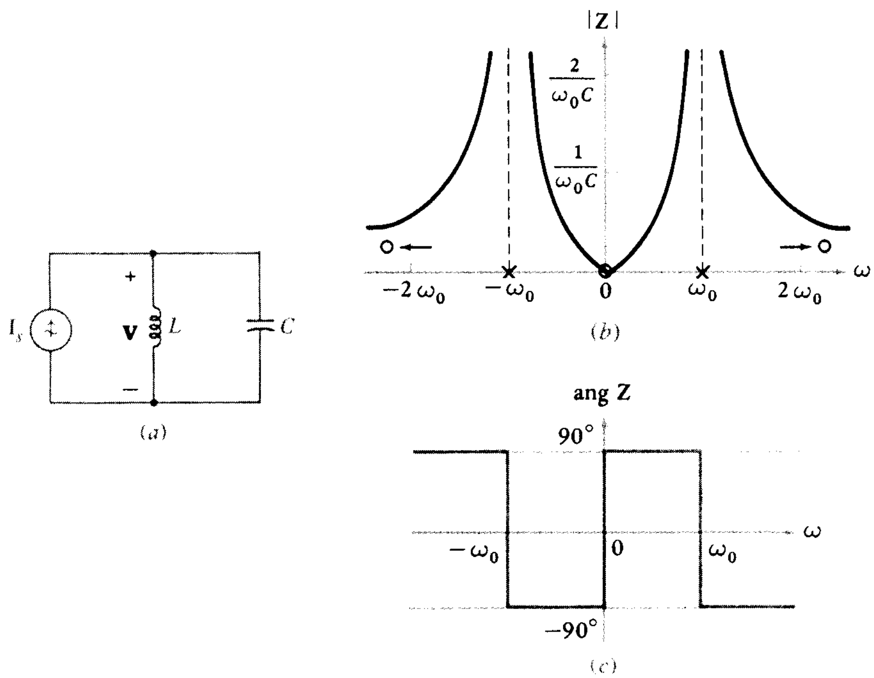
$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}_s} = \frac{L/C}{j(\omega L - 1/\omega C)}$$

or

$$\mathbf{Z} = -j\frac{1}{C}\frac{\omega}{\omega^2 - 1/LC} \tag{6}$$

Figure 9-14

(*a*) A sinusoidally excited parallel LC circuit. (*b*) The magnitude of the input impedance, $\mathbf{Z} = \mathbf{V}/\mathbf{I}_s$, and (*c*) the angle of the input impedance are plotted as functions of ω.



By letting

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and factoring the expression for the input impedance, the magnitude of the impedance may be written in a form which enables those frequencies to be identified at which the response is zero or infinite:

$$|\mathbf{Z}| = \frac{1}{C}\frac{|\omega|}{|(\omega - \omega_0)(\omega + \omega_0)|} \tag{7}$$

Such frequencies are termed *critical frequencies,* and their early identification simplifies the construction of the response curve. We note first that the response has zero amplitude at $\omega = 0$; when this happens, we say that the response has a *zero* at $\omega = 0$, and we also describe the frequency at which it occurs as a zero. Response of infinite amplitude is noted at $\omega = \omega_0$ and $-\omega_0$; these frequencies are called *poles,* and the response is said to have a pole at each of these frequencies. Finally, we note that the response approaches zero as $\omega \to \infty$, and thus $\omega = \pm\infty$ is also a zero.[2]

The locations of the critical frequencies should be marked on the ω axis, by using small circles for the zeros and crosses for the poles. Poles or zeros at infinite frequency should be indicated by an arrow near the axis, as shown in Fig. 9-14*b*. The actual drawing of the graph is made easier by adding broken vertical lines as asymptotes at each pole location. The completed graph of magnitude versus ω is shown in Fig. 9-14*b*; the slope at the origin is *not* zero.

An inspection of Eq. (6) shows that the phase angle of the input impedance must be either $+90°$ or $-90°$; no other values are possible, as must apparently be the case for any circuit composed entirely of inductors and capacitors. An analytical expression for ang **Z** would therefore consist of a series of statements that the angle is $+90°$ or $-90°$ in certain frequency ranges. It is simpler to present the information graphically, as shown in Fig. 9-14*c*. Although this curve is only a collection of horizontal straight line segments, errors are often made in its construction, and it is a good idea to make certain that it can be drawn directly from an inspection of Eq. (6).

Example 9-5 Construct plots of the amplitude and phase of $\mathbf{Z}_{in}$ versus ω for the *LC* circuit of Fig. 9-15.

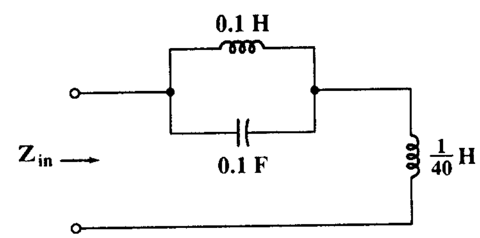


Figure 9-15

An *LC* circuit whose frequency response is determined in Example 9-5.

Solution: We begin by finding the analytical expression for $\mathbf{Z}_{in}$,

$$\mathbf{Z}_{in} = \frac{(j\omega/10)(10/j\omega)}{j\omega/10 + 10/j\omega} + \frac{j\omega}{40} = \frac{1}{(100-\omega^2)/j10\omega} + \frac{j\omega}{40}$$

$$= \frac{j10\omega}{100-\omega^2} + \frac{j\omega}{40} = j\omega\left[\frac{100-\omega^2+400}{40(100-\omega^2)}\right]$$

$$= \frac{j\omega}{40}\left[\frac{\omega^2-500}{\omega^2-100}\right]$$

Therefore,

$$\mathbf{Z}_{in} = j\,\frac{\omega(\omega-22.4)(\omega+22.4)}{40(\omega-10)(\omega+10)} \qquad (8)$$

[2] It is customary to consider plus infinity and minus infinity as being the same point. The phase angle of the response at very large positive and negative values of ω need not be the same, however.

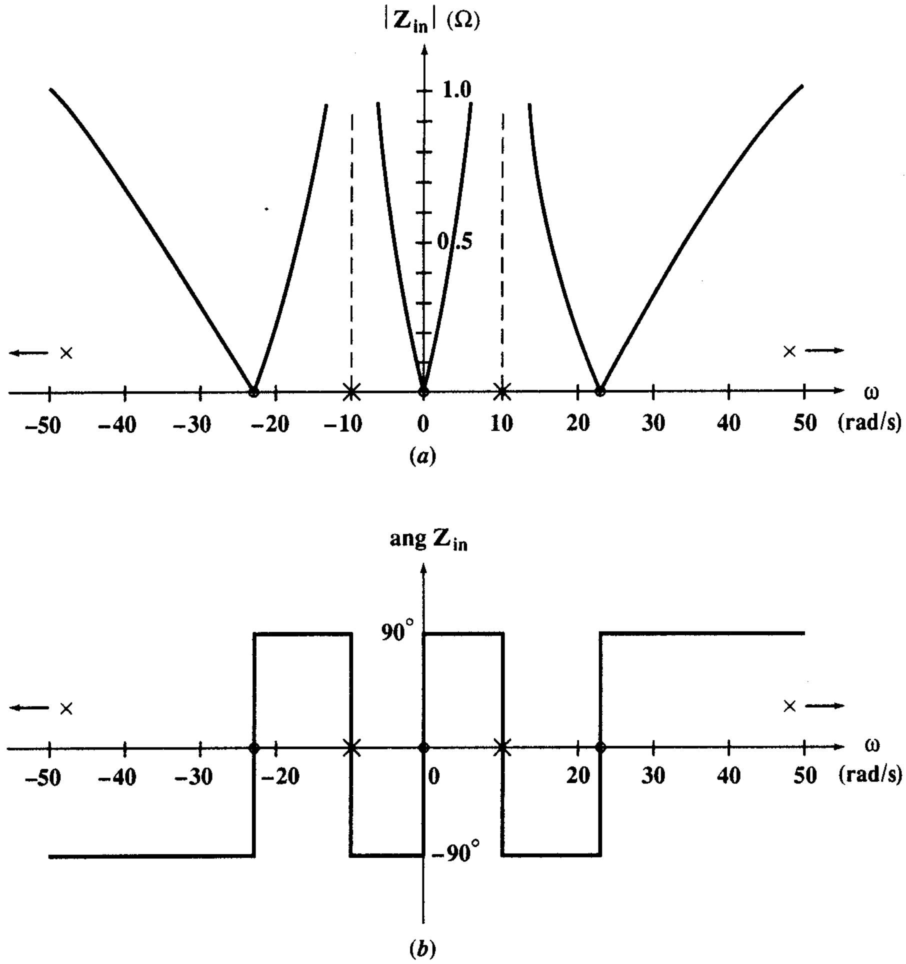
We now may write an expression for the magnitude of $\mathbf{Z}_{in}$,

$$|\mathbf{Z}_{in}| = \left|\frac{\omega(\omega - 22.4)(\omega + 22.4)}{40(\omega - 10)(\omega + 10)}\right|$$

We note the presence of zeros at $\omega = 0$, -22.4, and 22.4 rad/s; poles are found at $\omega = -10$ and 10 rad/s. Since $|\mathbf{Z}_{in}|$ approaches infinity as ω approaches infinity, there is also a pole at $\omega = \pm\infty$. These six critical frequencies are indicated on the sketch of Fig. 9-16a, and the response curve is roughed in.

Figure 9-16

For the LC network of Fig. 9-15, sketches are shown of (a) $|\mathbf{Z}_{in}|$ versus ω; (b) ang $\mathbf{Z}_{in}$ versus ω.

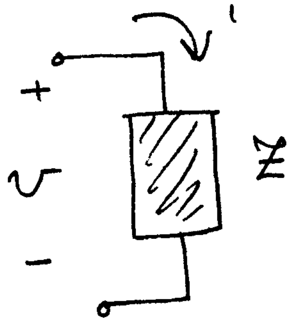


From Eq. (8), we can see that a large (in magnitude) negative value of ω, such as $\omega = -100$, produces a negative imaginary $\mathbf{Z}_{in}$, here $\mathbf{Z}_{in} = -j2.40\ \Omega$; hence the phase angle is $-90°$ for $-\infty < \omega < -22.4$. The angle alternates between $-90°$ and $+90°$ every time the frequency passes a pole or a zero. The resultant phase plot is shown as Fig. 9-16b. ∎

Drill Problems

9-7. For the circuit shown in Fig. 9-17, sketch, as a function of ω: (a) $|\mathbf{V}_1|$; (b) ang $\mathbf{V}_1$; (c) $|\mathbf{I}_2|$. *Ans:* $|\mathbf{V}_1(j2)| = 33.8$ V; ang $\mathbf{V}_1(j2) = -5.44°$; $|\mathbf{I}_2(j2)| = 0.358$ A

$\frac{E\,18}{143}$:



$$v(t) = 20\sqrt{2}\cos 120t$$

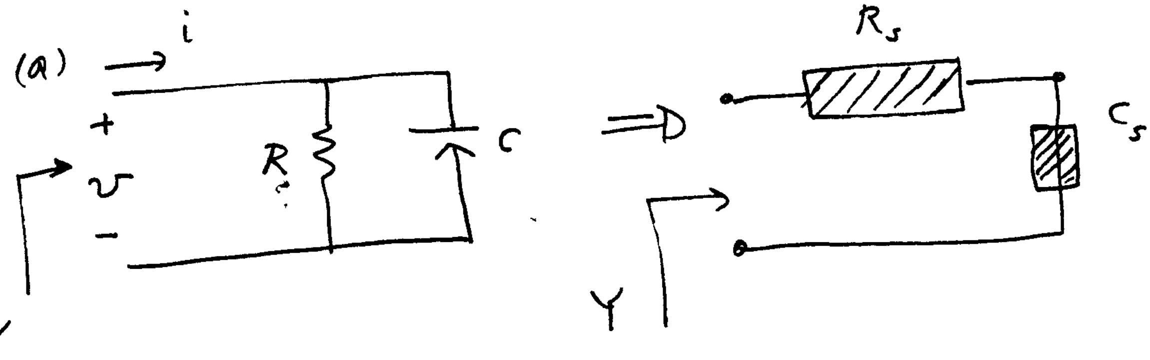
$$i(t) = 4\sqrt{2}\cos(120t + 37^\circ)$$

$$Z = \frac{V}{I} = \frac{20\angle 0}{4\angle 37} = 5\angle -37$$

$$= 5\cos(-37) + j\,5\sin(-37)$$

$$= 4 - j3 \;\Omega.$$

$\frac{E\,21}{143}$: (a)

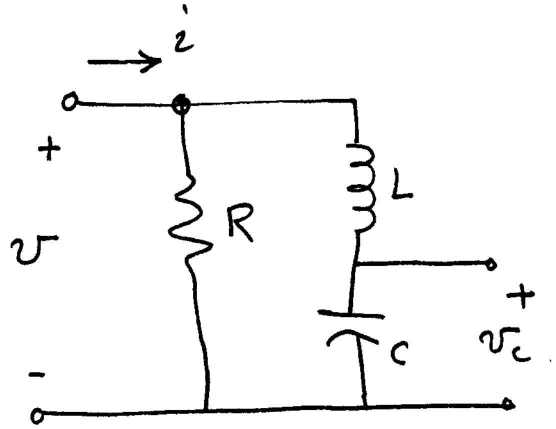


$$Y = \frac{1}{Z} = Y_R + Y_c = \frac{1}{R} + j\omega C = \frac{1}{2000} + j(1000)(2.5)\times 10^{-6}$$

For Series circuit $\Rightarrow$ $Y = \dfrac{1}{R_s + \dfrac{1}{j\omega C_s}} = \dfrac{\omega^2 C_s^2 R_s}{1+\omega^2 C_s^2 R_s^2} + j\dfrac{\omega C_s}{1+\omega^2 C_s^2 R_s}$

$\therefore$ $C_s = 2.6\,\mu F$, $R_s = 76.92\;\Omega$.

$\frac{E\ 24}{144}$:



$v = 5\sqrt{2} \cos 5t$

$R = 2\ \Omega$

$L = 1\ H$

$C = .05\ F.$

(a) $Y_{in}(j\omega) = ?$

$$Z(j\omega) = R \parallel \left(j\omega L + \frac{1}{j\omega C} \right) = \frac{R\left(j\omega L + \frac{1}{j\omega C}\right)}{R + j\omega L + \frac{1}{j\omega C}}$$

$$Y(j\omega) = \frac{1}{Z(j\omega)} \quad , \quad \omega = 5$$

$$= 1.12 \angle -63.4^\circ \ \ S$$

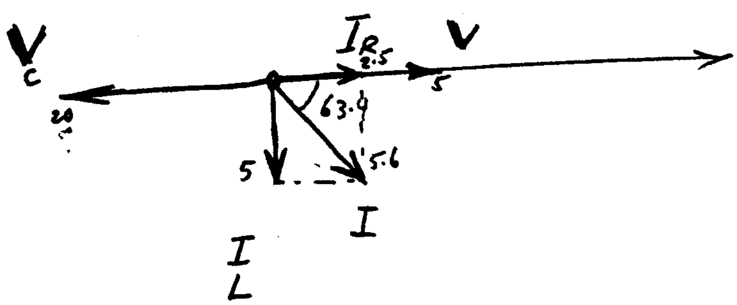
(b) $$I_R = \frac{V}{R} = \frac{5\angle 0}{2} = 2.5\angle 0 \text{ A.}$$

$$I = \frac{V}{Z} = VY = (5\angle 0)(1.12\angle -63.4^\circ)$$

$$= \underline{5.6\angle -63.4^\circ \text{ A}}$$

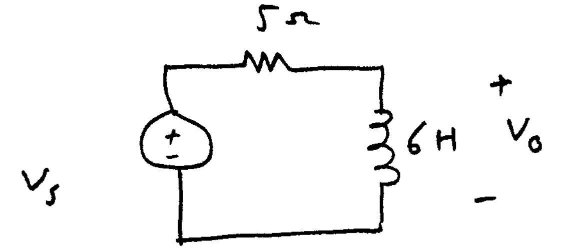
$$I_L = I - I_R = 5.6\angle -63.4^\circ - 2.5\angle 0^\circ = \underline{5\angle -90^\circ}$$

$$V_c = \frac{1}{j\omega c} I_L = 20\angle -180^\circ.$$



Exa.

4.4 : 272

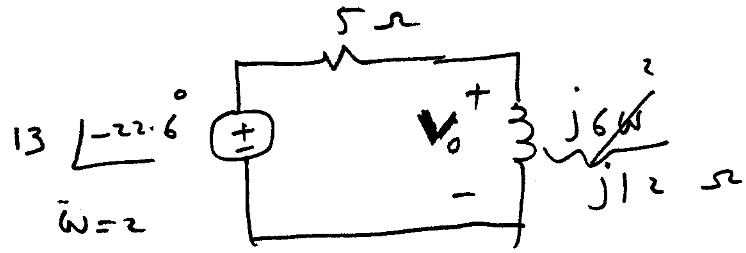


$\tan\theta$

$12\left(\cos 2t + \frac{5}{12}\sin 2t\right)$

$= \frac{12}{\cos\theta}\cos(2t -$

$v_s = 12\cos 2t + 5\sin 2t. = 13\cos(2t - 22.62^\circ)$

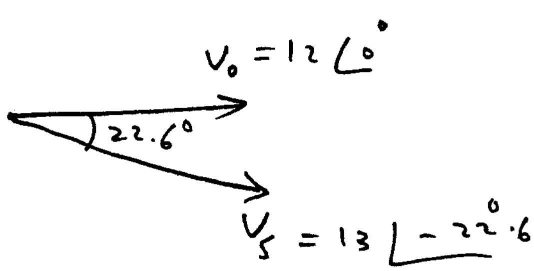


$$V_o = \frac{j12}{5+j12}\left(13\angle{-22.6^\circ}\right) = \frac{12\angle 90^\circ}{\cancel{13}\angle 67.3}\left(\cancel{13}\angle{-22.6}\right)$$

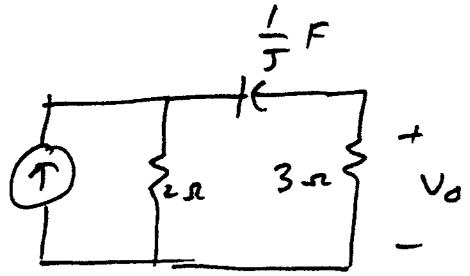
$$= 12\angle 0^\circ$$

$\therefore\ v_o(t) = 12\cos 2t$

the output leads the input by 22.62° and, hence, the circuit is a lead network.



4.5 / 222 :

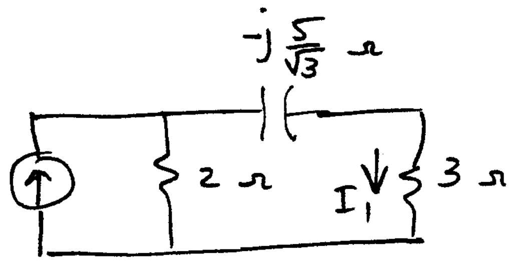


$8 \sin\sqrt{3}t$

$= 8\cos(\sqrt{3}t - 90^\circ)$ A

$8 \angle -90^\circ$

$\omega = \sqrt{3}$

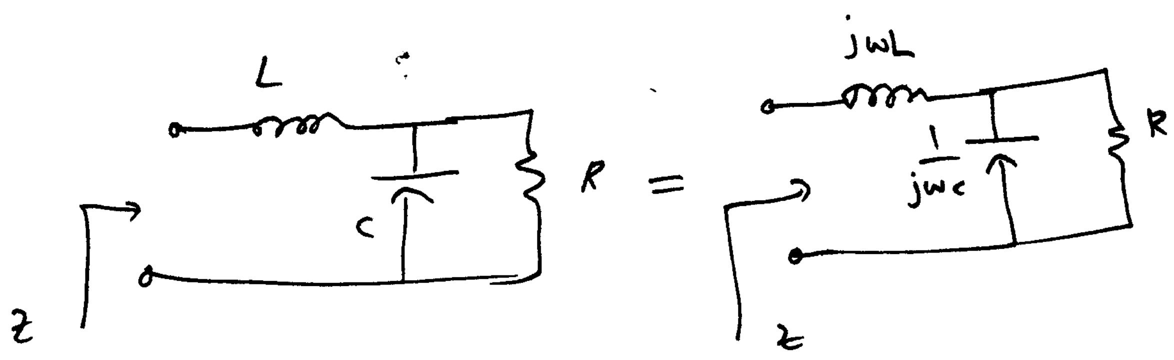


$$I_1 = \frac{2}{2+3-j\frac{5}{\sqrt{3}}}\left(8\angle -90^\circ\right) = \frac{16\angle -90^\circ}{5.774\angle -30^\circ} = 2.77\angle -60^\circ \text{ A}$$

$$V_o = 3I_1 = 8.314\angle -60^\circ \text{ V}$$

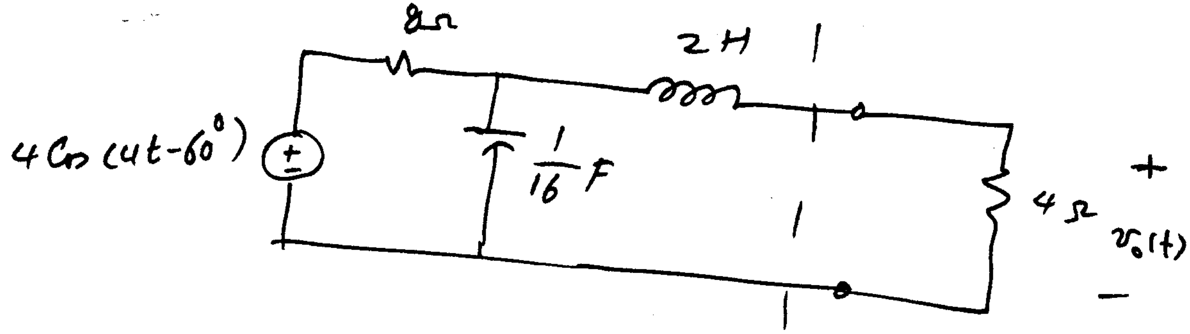
$$\therefore \quad v_o(t) = 8.314\cos(\sqrt{3}t - 60^\circ) \text{ V.}$$

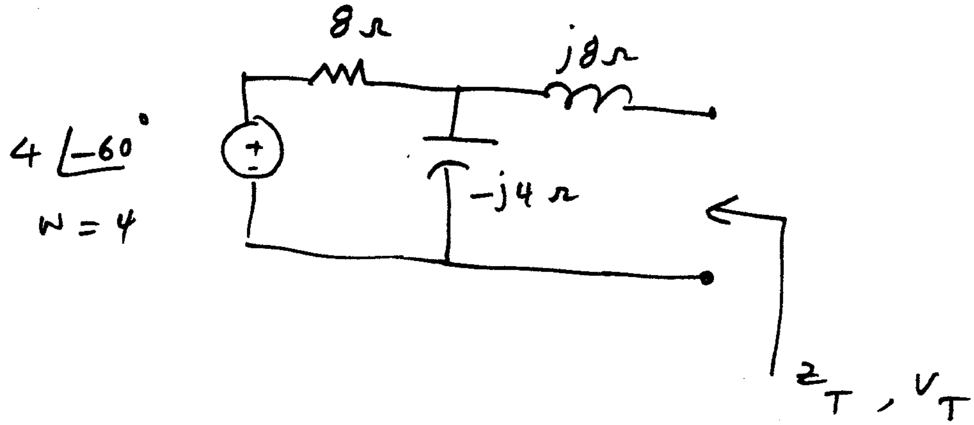
4.8 / 222 :



$$z = j\omega L + \left(\frac{1}{j\omega C} \,//\, R\right) = j\omega L + \frac{R\left(\frac{1}{j\omega C}\right)}{R + \frac{1}{j\omega C}}.$$

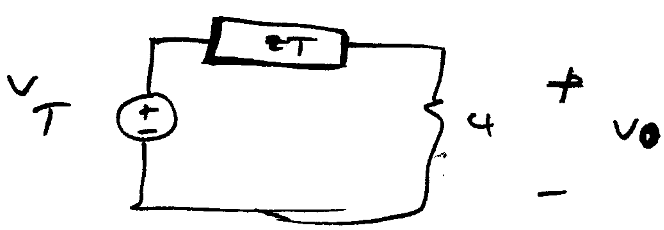
Exa





$$V_T = \frac{-j4}{8-j4}(4\angle -60^\circ) = \frac{4}{\sqrt{5}}\angle -123.43^\circ$$

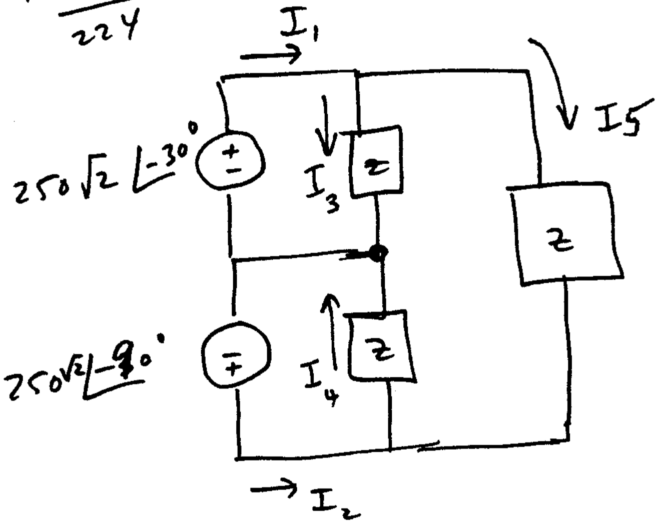
$$Z_T = 8 \,//\, (-j4) + j8 = \frac{8}{5} + j\frac{24}{5}\ \Omega$$



$$V_o = \frac{4}{4+Z_T} V_T = .97\angle -164.03^\circ$$

$$\therefore\ v_o(t) = .97\cos(4t - 164^\circ)$$

$\# \frac{4.15}{224}$:



$z = 78 - j45\ \Omega$

$$I_3 = \frac{250\sqrt{2}\angle{-30^\circ}}{z}$$

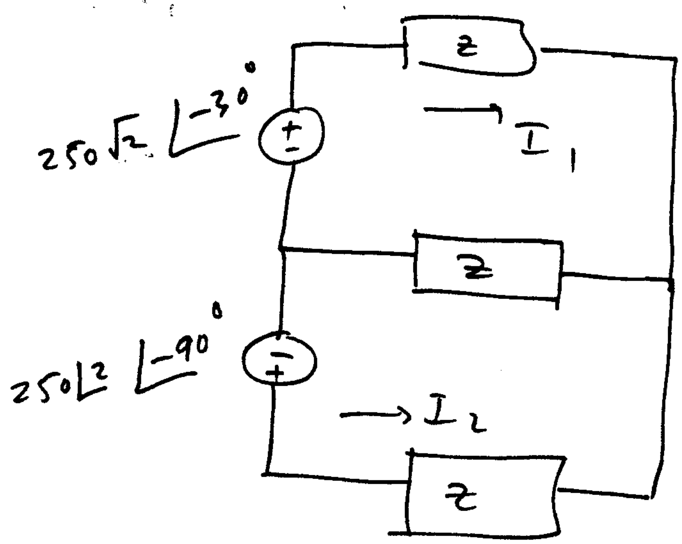
$$I_4 = \frac{250\sqrt{2}\angle{-30^\circ}}{z}$$

$$I_5 = \frac{250\sqrt{2}\angle{-30^\circ} - 250\sqrt{2}\angle{-90^\circ}}{z}$$

$$I_1 = I_3 + I_5$$

$$I_2 = I_4 - I_5$$

Exa



$Z = 26 - j15$;

$$-250\sqrt{2}\angle -30^\circ + I_1 Z + (I_1 + I_2) Z = 0$$

$$-250\sqrt{2}\angle -90^\circ + I_2 Z + (I_1 + I_2) Z = 0$$

$\therefore$ $I_1 = 6.8\angle 30^\circ$; A

$I_2 = 6.8\angle -90^\circ$; A

Remarks

If $\quad v = v_1 + v_2 + \cdots + v_n$

where all the voltages are sin fac's of the same freq,

then

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots\cdots + \mathbf{V}_m .$$

Pf. Let $\quad v_1(t) = V_{m_1} \cos(\omega t + \phi_1) = \mathrm{Re}\{V_{m_1} e^{j\phi_1} \cdot e^{j\omega t}\}$

$$v_2(t) = V_{m_2} \cos(\omega t + \phi_2) = \mathrm{Re}\{V_{m_2} e^{j\phi_2} \cdot e^{j\omega t}\}.$$

$$\underbrace{v(t)}_{\mathrm{Re}\{V_m e^{j\phi} \cdot e^{j\omega t}\}} = v_1(t) + v_2(t) = \mathrm{Re}\left\{\left(V_{m_1} e^{j\phi_1} + V_{m_2} e^{j\phi_2}\right) \cdot e^{j\omega t}\right\}$$

$$\therefore \quad V_m e^{j\phi} = V_{m_1} e^{j\phi_1} + V_{m_2} e^{j\phi_2}$$

$$\therefore \quad \mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$$

Exa $\frac{9.5}{367}$

$$y_1(t) = 20 \cos(\omega t - 30^\circ)$$

$$y_2(t) = 40 \cos(\omega t + 60^\circ)$$

Find $y(t) = y_1(t) + y_2(t)$. We use

$$Y = Y_1 + Y_2 = 20 \angle -30^\circ + 40 \angle 60^\circ$$

$$= 20(\cos(+30) - j \sin(30^\circ)) + 40(\cos 60^\circ + j \sin 60^\circ)$$

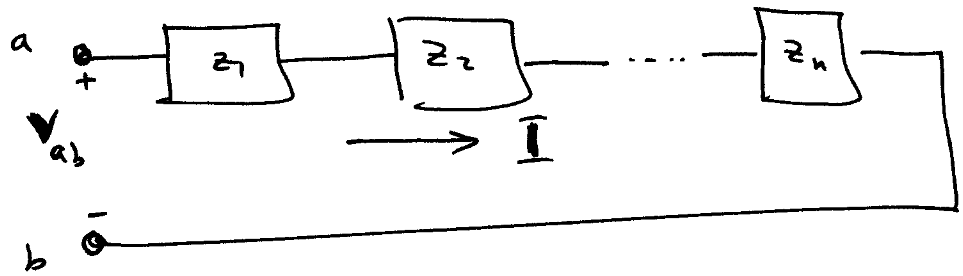
$$= (17.32 - j10) + (20 + j34.64) = 37.32 + j24.64$$

$$= 44.72 \angle 33.43^\circ$$

$$y(t) = \mathcal{P}^{-1}\{44.72 e^{j33.43^\circ}\} = 44.72 \cos(\omega t + 33.43^\circ).$$

Remarks

1

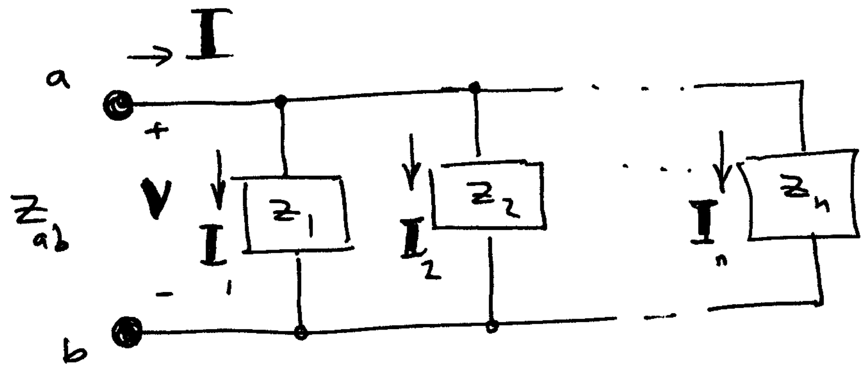


$$V_{ab} = Z_1 I + Z_2 I + \cdots + Z_n I$$

$$= (Z_1 + Z_2 + \cdots + Z_n) I$$

$$\therefore Z_{ab} = \frac{V_{ab}}{I} = Z_1 + Z_2 + \cdots + Z_n$$

②



$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n$$

$$\frac{\mathbf{V}}{Z_{ab}} = \frac{\mathbf{V}}{Z_1} + \frac{\mathbf{V}}{Z_2} + \cdots + \frac{\mathbf{V}}{Z_m}$$

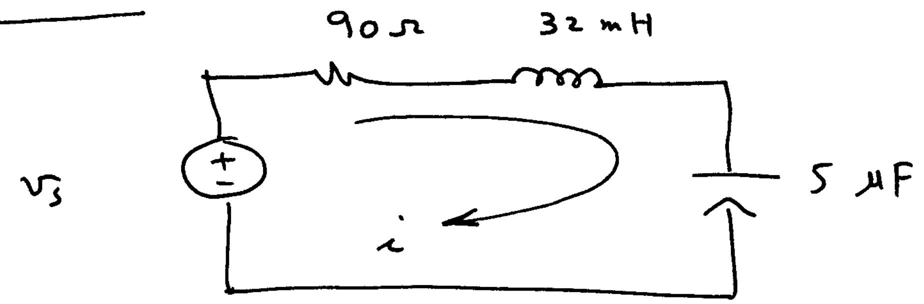
$$\therefore \frac{1}{Z_{ab}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$$

or

$$Y_{ab} = Y_1 + Y_2 + \cdots + Y_n$$

siemens

Exa $\frac{9.6}{375}$:



$v_s(t) = 750 \cos(5000t + 30^\circ)$ (*)

a) Construct the freq-domain equivalent circuit.

b) Calculate the ss current i by phasor method.

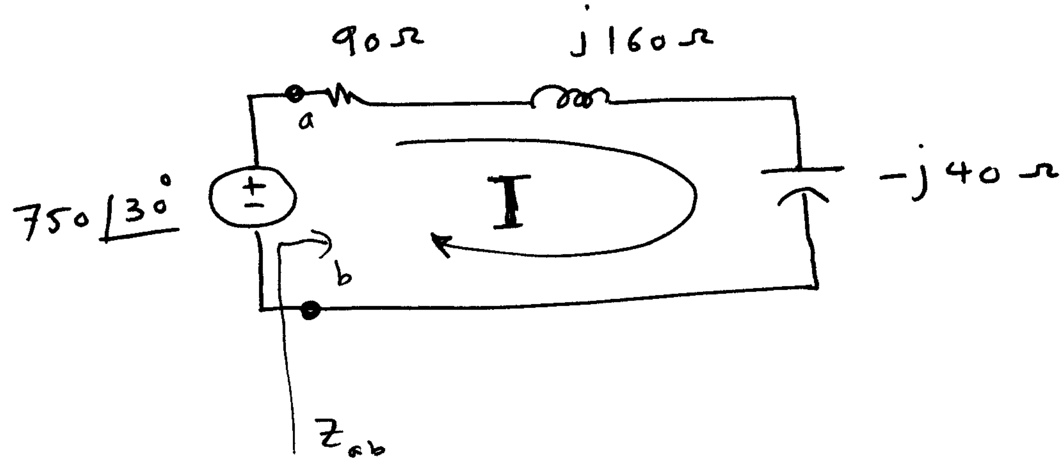
Sol^n:

(*) $\Rightarrow$ $\omega = 5000$ rad/s.

$$Z_L = j\omega L = j(5000)(32 \times 10^{-3}) = j160\ \Omega$$

$$Z_c = \frac{1}{j\omega c} = -j\frac{1}{(5000)(5 \times 10^{-6})} = -j40\ \Omega$$

$$V_s = 750 \angle 30^\circ$$



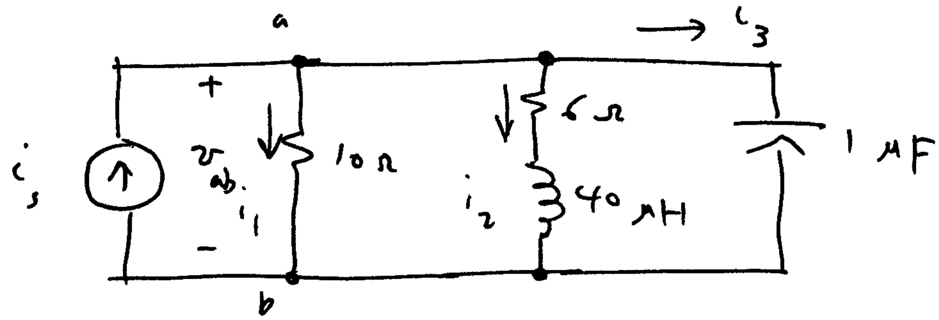
b)

$$Z_{ab} = 90 + j160 - j40 = 90 + j120 = 150 \angle 53.13^\circ \ \Omega$$

$$\mathbf{I} = \frac{750 \angle 30^\circ}{150 \angle 53.13} = 5 \angle -23.13^\circ \ \text{A.}$$

$$i_{ss}(t) = 5 \cos(5000t - 23.13^\circ) \ \text{A.}$$

Exa 9.7
377

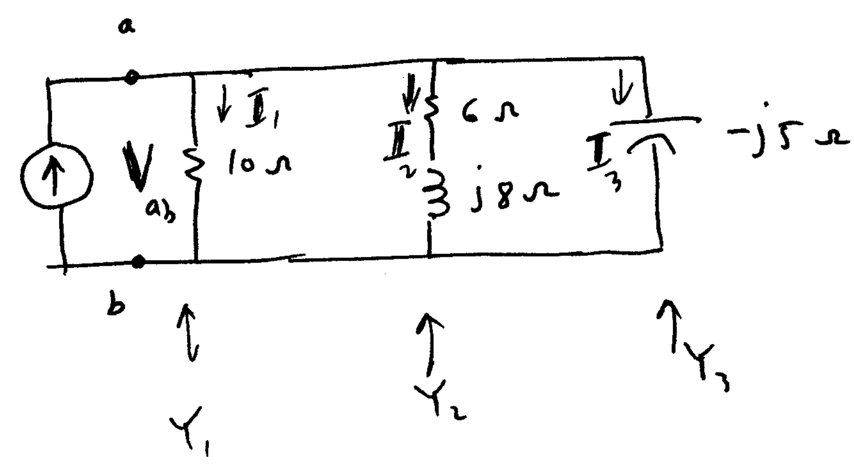


$i_s(t) = 8\cos(200{,}000t)$ A.

$\omega = 200{,}000$ rad/s.

$$Z_l = j\omega L = j(200{,}000)(40 \times 10^{-6}) = j8\ \Omega.$$

$$Z_c = \frac{1}{j\omega c} = -j\frac{1}{(200{,}000)(10^{-6})} = -j5\ \Omega.$$



$$Y_{ab} = Y_1 + Y_2 + Y_3$$

$$Y_1 = \frac{1}{10} \quad S$$

$$Y_2 = \frac{1}{Z_2} = \frac{1}{6+j8} = \frac{6-j8}{\cancel{36+64}_{100}} = 0.06 - j0.08 \quad S$$

$$Y_3 = \frac{1}{Z_3} = \frac{1}{-j5} = j0.2 \quad S$$

$$\therefore \; Y_{ab} = \underbrace{\cancel{\frac{1}{10}}}_{0.1} + (0.06 - j0.08) + (j0.2)$$

$$= 0.16 + j0.12 = 0.2\angle 36.87^\circ \quad S.$$

$$Z_{ab} = \frac{1}{Y_{ab}} = \frac{1}{0.2\angle 36.87^\circ} = 5\angle -36.87^\circ.$$

$$V_{ab} = Z_{ab}\, I = (5\angle -36.87^\circ)(8\angle 0) = 40\angle -36.87^\circ$$

V.

$$\mathbf{I}_1 = \frac{\mathbf{V}_{ab}}{10} = 4\angle{-36.87^\circ} \quad A$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_{ab}}{6+j8} = \frac{40\angle{-36.87^\circ}}{10\angle} = 4\angle{-90^\circ} \quad A$$

$$\mathbf{I}_3 = \frac{\mathbf{V}_{ab}}{5\angle{-90^\circ}} = \frac{40\angle{-36.87^\circ}}{5\angle{-90}} = 8\angle{53.13^\circ} \quad A$$

or

$$v_{ab}(t) = 40 \cos(200{,}000t - 36.87^\circ) \quad V$$

$$i_1(t) = 4 \cos(200{,}000t - 36.87^\circ) \quad A$$

$$i_2(t) = 4 \cos(200{,}000t - 90^\circ) \quad A$$

$$i_3(t) = 8 \cos(200{,}000t + 53.13^\circ) \quad A$$

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